L27-Operators: Stretches

Monday, September 21, 2015 6:36 AM

* Geometry of Stretches & Rotations

$Sv = \lambda v$

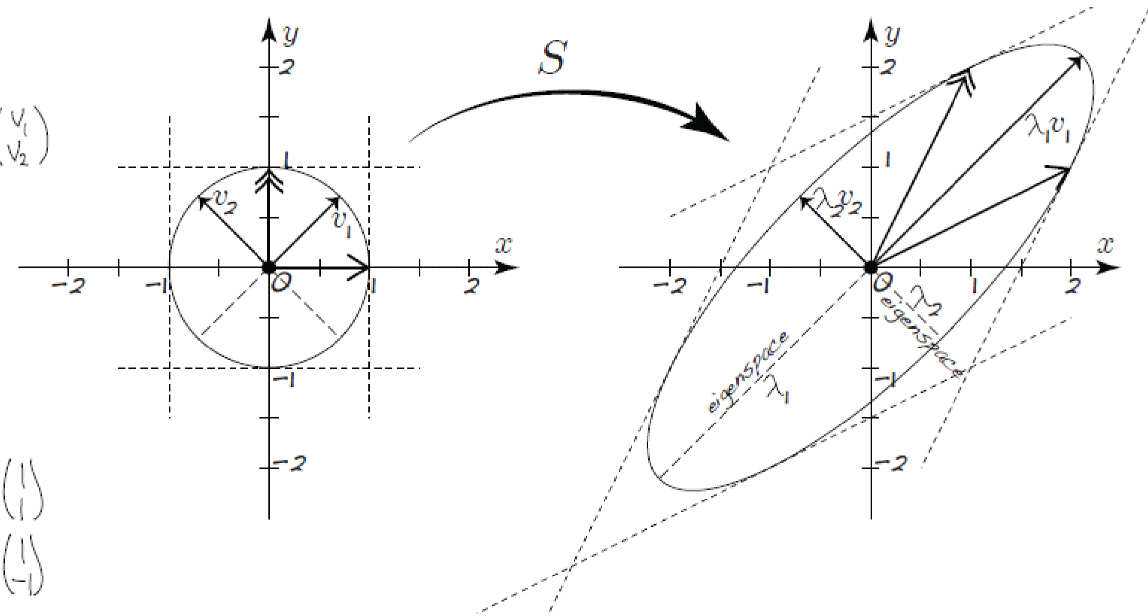
$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

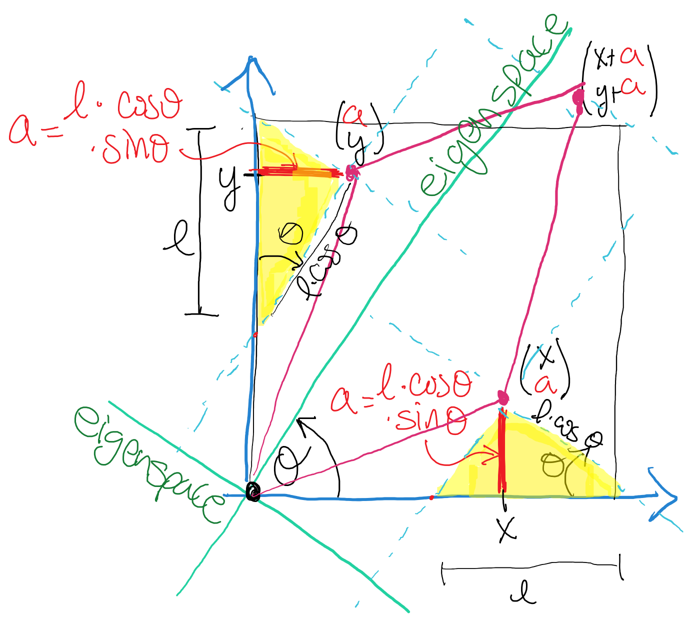
$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 1\begin{pmatrix} 1 \\ -1 \end{pmatrix}$



- principle axes of an ellipse are normal (perpendicular)

$$(x-x_0,\ y-y_0)\begin{pmatrix} \frac{1}{a^2} & \varepsilon \\ \varepsilon & \frac{1}{b^2} \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix} = 1$$

- symmetric matrices are pure stretches



$$M = \begin{pmatrix} x & a \\ a & y \end{pmatrix}$$

- any shape of trapezoid can be made by flattening a square λ along θ line

- we still need to rotate to get an arbitrary trapezoid $M = RS$

* Eigensystem notation

- if $M^\dagger = M$ (Hermitian) then there exists a complete set of eigenvalues λ_i and eigenvectors $\vec{v}_i$ such that $M\vec{v}_i = \lambda_i \vec{v}_i$ and $\vec{v}_i^\dagger v_j = \delta_{ij}$.

- we can put $\vec{v}_i$ together to form a unitary transformation matrix $V = (\vec{v}_1 | \vec{v}_2 | \cdots \vec{v}_n)$,

$$\text{so} \quad MV = VD \qquad D = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix}$$ (diagonalization)

$$\begin{pmatrix} m_{xx} & m_{xy} \\ m_{yx} & m_{yy} \end{pmatrix} \begin{pmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{pmatrix} = \begin{pmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$$

- V is unitary (orthonormal): $V^\dagger V = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix}_{\circ} \cdot (\vec{v}_1 \, \vec{v}_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

- V is also closed: $VV^\dagger = (\vec{v}_1 \, \vec{v}_2) \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \end{pmatrix}_{\circ} = \underbrace{\vec{v}_1 \vec{v}_1^{\circ}}_{P_1} + \underbrace{\vec{v}_2 \vec{v}_2^{\circ}}_{P_2} = 1$

- D = matrix elements of M in basis V: $\boxed{\begin{matrix} D = V^\dagger M V \\ I = V^\dagger V \end{matrix}} = \langle v | M | v \rangle$

- eigenvalue decomposition of M: $\boxed{\begin{matrix} M = VDV^\dagger = \sum_i \lambda_i P_i \\ 1 = VV^\dagger = \sum_i P_i \end{matrix}}$

* Characterization of linear operators

a) ANY linear function has a singular value decomposition (SVD)

$$M = R\overset{\text{symmetric}}{S} = R\,\overset{\text{eig.}}{VWV^T} = UWV^T \quad \text{where} \quad U^TU = I_B$$

$$\underset{A\to B}{M} = \underset{\text{rotation}}{R}\,\overset{\text{symmetric}}{S} = \underbrace{R\,V}_{\text{2 rotations}}\overbrace{\underset{\text{diagonal}}{W}\,V^T}^{\text{eig.}} = \underbrace{U}_{\text{combined rotation}} W V^T \quad \text{where} \quad \begin{array}{l} U^T U = I_B \\ V^T V = I_A \\ W \text{ diagonal} \end{array}$$

eigenvalues λ_i of W: singular values

V, U are eigen-basis of A, B respectively.

b) ANY linear operator has n complex eigenvalues: λ_i
(Fundamental Theorem of Algebra on characteristic equation)

γ_i = geometric multiplicity = number of each eigenvalue

μ_i = algebraic multiplicity = dimension of eigenspace

- if $\mu_i = \gamma_i$ then the matrix is diagonalizable
- otherwise it still has a Jordan decomposition $M = V \begin{pmatrix} \lambda_1 & 1 & & \\ & \lambda_1 & & \\ & & \lambda_2 & 1 \\ & & & \lambda_2 \ddots \end{pmatrix} V^{-1}$
  note that V is not necessarily orthogonal or normal! $\to V^{-1}$, not V^T
- this is a similarity transform

c) ANY normal matrix M (ie. $M^\dagger M = M M^\dagger$)
has a unitary diagonalization
(orthonormal eigenvectors). $V^\dagger V = I$

d) [anti] Hermitian operators have [imaginary] real eigenvalues.

$$\lambda_i^* v_i^\dagger v_j = v_i^\dagger H^\dagger v_j = v_i^\dagger H v_j = v_i^\dagger v_j \lambda_j$$

if $i=j$: $(\lambda_i - \lambda_i^*) v_i^\dagger v_i = 0 \to \lambda_i$ real

if $\lambda_i \neq \lambda_j$: $(\lambda_i - \lambda_j^*) v_i^\dagger v_j = 0 \to v_i^\dagger v_j = 0$ (unitary)

* what does $\dagger$ have to do with operators?

Symmetric / antisymmetric vs. Symmetric / orthogonal decomposition

~ recall complex numbers $u = \rho + i\phi \qquad \rho^* = \rho \qquad (i\phi)^* = -i\phi$

$$e^{u} = e^{\rho + i\phi} = r e^{i\phi} \qquad |e^{i\phi}|^2 = e^{-i\phi} e^{i\phi} = e^{i0} = 1$$

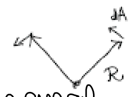
~ similar behaviour of symmetric / antisymmetric matrices

M artibrary matrix
T symmetric
A antisymmetric

S symmetric
R orthogonal

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & (b+c)/2 \\ (b+c)/2 & d \end{pmatrix} + \begin{pmatrix} 0 & (b-c)/2 \\ (c-b)/2 & 0 \end{pmatrix} = T + A$$

$$K = e^{M} = 1 + M + \tfrac{1}{2!}M^2 + \tfrac{1}{3!}M^3 + \ldots \quad = e^{T+A} \neq e^{T} e^{A} \text{ in general.}$$



$$S = e^{T} = e^{VWV^{-1}} = V e^{W} V^{-1} \qquad R = e^{A} \qquad R^T R = (e^{A})^T e^{A} = e^{A^T + A} = e^{0} = I$$

$$\det \begin{pmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & \ddots \end{pmatrix} = e^{\lambda_1} \cdot e^{\lambda_2} \cdots = e^{\lambda_1 + \lambda_2 + \ldots} = e^{\mathrm{tr} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \end{pmatrix}} \qquad \det e^{A} = e^{\mathrm{tr}\, A} = e^{0} = 1$$

if $e^{T+A} = e^{T} \cdot e^{A}$ [ie T, A commute], then

$K = S \cdot R$ polar decomposition