L30-Hilbert Space

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* Hilbert space: infinite-dimensional vector space where all inner products converge

[where $\langle \phi | \psi \rangle = \sum_{n=0}^{\infty} a_n^* b_n$ or $\langle f | g \rangle = \int f^*(x) g(x) dx$ always finite]

Most theorems carry over from finite-dimensional spaces.

* Everything follows from the following principles & notation

	finite	Hilbert	functional
orthonormality	$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$	$\langle n \vert m \rangle = \delta_{nm}$	$\langle x \vert x' \rangle = \delta(x-x')$
closure	$\sum \hat{e}_i \hat{e}_i \cdot = I$	$\sum_n \vert n \rangle\langle n \vert = I$	$\int dx \vert x \rangle\langle x \vert = I$
components	$v_i = \hat{e}_i \cdot \vec{v}$	$c_n = \langle n \vert \psi \rangle$	$f(x) = \langle x \vert f \rangle$
matrix elements	$M_{ij} = \hat{e}_i \cdot M(e_j)$	$H_{mn} = \langle m \vert H \vert n \rangle$	$G(x,x') = \langle x \vert G \vert x' \rangle$

* Correspondence of Linear Algebra principles:

	$\mathbb{R}^n$ Finite Vectors	(discrete infinite) $\mathcal{H}$ Hilbert space.	(continuous infinite) $\mathcal{F}$ functional space
vector basis component	$\vec{V} = \hat{e}_i V_i$ $\begin{pmatrix} x \\ x \\ x \end{pmatrix}$	$\vert \psi \rangle = \sum_{n=0}^{\infty} \vert \phi_n \rangle c_n$	$\vert f \rangle = \int dx \vert x \rangle f(x)$
inner product	$\vec{V} \cdot \vec{W} = V_i^* W_i$ $(x\,x\,x)\begin{pmatrix} x \\ x \\ x \end{pmatrix}$	$\langle \psi \vert \chi \rangle = \sum_{n=0}^{\infty} c_n^* d_n$	$\langle f \vert g \rangle = \int dx\, f^*(x) g(x)$
adjoint (dual) vector	$\vec{V} \cdot = V_i^* \hat{e}_i \cdot$ $(x\,x\,x)$	$\langle \psi \vert = \sum_{n=0}^{\infty} c_n^* \langle \phi_n \vert$	$\langle f \vert = \int dx\, f^*(x) \langle x \vert$
orthonormal (unitary) basis.	$\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$	$\langle \phi_i \vert \phi_j \rangle = \delta_{ij}$	$\langle k \vert k' \rangle = \delta(k-k')$
completeness (closure) expansion	$\sum \hat{e}_i \hat{e}_i \cdot = I$	$\sum_n \vert \phi_i \rangle\langle \phi_i \vert = I$	$\int dk \vert k \rangle\langle k \vert = I$
operator components	$\vec{w} = A \vec{v}$ $w_i = A_{ij} V_j$	$\vert \chi \rangle = A \vert \psi \rangle$ $d_m = A_{mn} c_n$	$\vert g \rangle = A \vert f \rangle$ $g(x) = \int dx' A(x,x') f(x)$
Hermitian adjoint	$(A\vec{v})^\dagger = \vec{v}^\dagger A^\dagger$ $A^\dagger_{ij} = A^*_{ji}$	$\langle A \psi \vert = \langle \psi \vert A^\dagger = (A \vert \psi \rangle)^\dagger$ $\langle \psi \vert A \vert \chi \rangle^* = \langle \chi \vert A^\dagger \vert \psi \rangle$	
identity transform	$V_i = R_{ij'} V_{j'}$	$\psi(x) = \sum_n \phi_n(x)\, c_n$	$\psi(x) = \int dk \frac{1}{\sqrt{2\pi}} e^{ikx} A(k)$

identity transform	$V_i = R_{ij'} V_{j'}$ $R_{ij'} = \hat{e}_i \cdot e_{j'}$	$\Psi(x) = \sum_n \phi_n(x)\, C_n$ $\phi_n(x) = \langle x \vert \phi_n \rangle$	$\Psi(x) = \int dk\, \frac{1}{\sqrt{2\pi}} e^{ikx} A(k)$ $\frac{1}{\sqrt{2\pi}} e^{ikx} = \langle x \vert k \rangle$
orthogonal transform	$R^\dagger_{i'k} R_{kj'} = \hat{e}_{i'} \cdot \hat{e}_j = \delta_{ij}$ $R^\dagger = R^{-1}$	$\sum_n \phi_n^*(x) \phi(x') = \delta(x-x')$ $\int dx\, \phi_m^*(x)\, \phi_n(x) = \delta_{mn}$	$\frac{1}{2\pi} \int dk\, e^{-ikx'} e^{ikx} = \delta(x-x')$
Hermitian transform	$A^\dagger = A \quad Av = \lambda v$ $V^\dagger V = I \quad V^\dagger A V = D$	$\mathcal{H}^\dagger = \mathcal{H} \quad \mathcal{H} \vert \phi_n \rangle = E \vert \phi_n \rangle$ $\langle \phi_n \vert \phi_m \rangle = \delta_{nm} \quad \mathcal{H} = \sum_n \vert \phi_n \rangle E_n \langle \phi_n \vert$	$p = -i\hbar \partial_x \quad p^\dagger = p \quad p \vert k \rangle = \hbar k \vert k \rangle$ $\langle k' \vert k \rangle = \delta(k-k') \quad p = \int dk \vert k' \rangle \hbar k \langle k \vert$