L33-Uncertainty Principle: Position

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* recall:

a) Heisenberg uncertainty principle: $\Delta x \cdot \Delta k \geq \frac{1}{2}$
complementarity between wavelength and position
"x" and "p" representations Fourier transforms

b) Simultaneous diagonalization of commuting operators
A & B have the same definite states $\Leftrightarrow$ $[A,B]=0$

$$[x,p]\psi = x(-i\hbar\partial_x)\psi + i\hbar\partial_x(x\psi) = \left(i\hbar\frac{\partial x}{\partial x}\right)\psi = i\hbar\psi$$

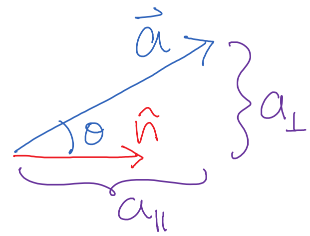
– connection between these two: Generalized Uncertainty Principle

* Projections – Schwartz inequality.

$$\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp} = \hat{n}\hat{n}\cdot\vec{a} - \hat{n}\times(\hat{n}\times\vec{a})$$

$$\vec{a}\cdot\left[(b^2\vec{a}) = b^2(a_{\parallel} + a_{\perp}) = \vec{b}\vec{b}\cdot\vec{a} - \vec{b}\times(\vec{b}\times\vec{a})\right]$$

$$a^2b^2 = a^2b^2(\cos^2\theta + \sin^2\theta) = (\vec{a}\cdot\vec{b})^2 + (\vec{a}\times\vec{b})^2$$



$$P_{\parallel b} = \frac{\vec{b}\vec{b}\cdot}{\vec{b}\cdot\vec{b}} \qquad P_{\parallel b}^2 = \frac{\vec{b}\vec{b}\cdot}{\vec{b}\cdot\vec{b}}\,\frac{\vec{b}\vec{b}\cdot}{\vec{b}\cdot\vec{b}} = \frac{\vec{b}\vec{b}\cdot}{\vec{b}\cdot\vec{b}} = P_{\parallel b}$$

$$P_{\perp b} = 1 - P_{\parallel b} \qquad P_{\perp b}^2 = (1-P_{\parallel b})^2 = 1 - 2P_{\parallel b} + P_{\parallel b}^2 = 1 - P_{\parallel b} = P_{\perp b}$$

– projection operators are idempotent:
only the first operation has any effect;
acts like the identity after the first

– compare nilpotent: $N^n = 0$ for some n
example: a_- on a finite space of states.

$$\langle f|f\rangle\langle g|g\rangle = \langle f|f\rangle\langle g|P_{\parallel f} + P_{\perp f}|g\rangle$$

$$= \langle f|g\rangle\langle g|f\rangle + \langle f|f\rangle\left|\underbrace{\left(1 - \frac{|f\rangle\langle f|}{\langle f|f\rangle}\right)}_{P_{\perp f}}|g\rangle\right|^2$$

$$= \langle f'|g\rangle\langle g|f'\rangle + \langle f|f\rangle \Big| \big(\underbrace{1 - \tfrac{|f\rangle\langle f|}{\langle f|f\rangle}}_{P_{\perp f}}\big)|g\rangle \Big|^2$$

$$\geq |\langle f|g\rangle|^2 \quad \text{equality if } |f\rangle = c|g\rangle \quad \text{ie. } P_{\perp f}|g\rangle = 0.$$

* uncertainty and operators:

$$\sigma_A^2 = \langle (\hat{A} - \langle A\rangle)^2\rangle = \langle \hat{A}^2 - 2A\langle A\rangle + \langle A\rangle^2\rangle = \langle \hat{A}^2\rangle - \langle \hat{A}\rangle^2$$

$$= \langle f|f\rangle \quad \text{where } |f\rangle = (\hat{A} - \langle\hat{A}\rangle)|\Psi\rangle$$

in a sense, $\langle \dots \rangle I$ is also idempotent!

$$\sigma_B^2 = \langle g|g\rangle \quad \text{where } |g\rangle = (\hat{B} - \langle\hat{B}\rangle)|\Psi\rangle$$

$$\sigma_A^2\sigma_B^2 = \langle f|f\rangle\langle g|g\rangle \geq |\underbrace{\langle f|g\rangle}_{z}|^2 \geq \Big[\tfrac{1}{2i}\underbrace{(\langle f|g\rangle - \langle g|f\rangle)}_{\text{imaginary part of } z}\Big]^2$$

$$\text{where } |z|^2 = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2 = \left(\tfrac{1}{2}(z+z^*)\right)^2 + \left(\tfrac{1}{2i}(z-z^*)\right)^2$$

$$z \equiv \langle f|g\rangle = \langle\Psi|(\hat{A} - \langle\hat{A}\rangle)(\hat{B} - \langle\hat{B}\rangle)|\Psi\rangle = \langle\Psi|\hat{A}\hat{B}|\Psi\rangle - \langle A\rangle\langle B\rangle$$

$$z^* = \langle g|f\rangle = \dots B \dots\dots A \dots\dots = \langle\Psi|\hat{B}\hat{A}|\Psi\rangle - \langle A\rangle\langle B\rangle$$

$$\operatorname{Re} z = \tfrac{1}{2}\langle\Psi|\{A,B\}|\Psi\rangle - \langle A\rangle\langle B\rangle \qquad \{A,B\} \equiv AB + BA$$

$$\operatorname{Im} z = \tfrac{1}{2i}\langle\Psi|[A,B]|\Psi\rangle \qquad [A,B] \equiv AB - BA$$

$$\text{thus} \quad \sigma_A^2\sigma_B^2 \geq \left(\tfrac{1}{2i}\langle[A,B]\rangle\right)^2 \qquad \sigma_x\sigma_p \geq \tfrac{1}{2i}\langle[\hat{x},\hat{p}]\rangle = \tfrac{i\hbar}{2i} = \tfrac{\hbar}{2}$$

* minimum uncertainty packet: $\sigma_x\sigma_p = \hbar/2$ (gaussian)

need 2 equalities: a) Schwartz $|f\rangle = c|g\rangle$; b) Im part $i\langle f|g\rangle \in \mathbb{R}$ $\Big\}$ $|f\rangle = ia|g\rangle$

for position & momentum space: $\hat{A} = \hat{p} = -i\hbar \frac{d}{dx}$ $\hat{B} = x$

$$\left(-i\hbar \frac{d}{dx} - \langle p \rangle\right)\psi(x) = ia\left(x - \langle x \rangle\right)\psi(x)$$

$$-i\hbar\, \psi' = \left(ia\left(x - \langle x \rangle\right) + \langle p \rangle\right)\psi$$

let

$$\xi = \frac{-a}{\hbar}\left(x - \langle x \rangle\right) + \frac{i}{\hbar}\langle p \rangle$$

$$d\xi = \frac{-a}{\hbar}\, dx$$

$$d \ln \psi = \frac{d\psi}{\psi} = \xi\, dx = d\, \frac{-\hbar}{2a}\xi^2$$

$$\psi(x) = \psi_0 e^{-\frac{\hbar}{2a}\xi^2} = A\underbrace{e^{-\frac{a}{2\hbar}(x - \langle x \rangle)^2}}_{\text{packet}}\underbrace{e^{i\langle p \rangle x/\hbar}}_{\text{phase (carrier)}}$$