L34-Uncertainty Principle: Time

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* time dependence of expected value:

$$\frac{d}{dt}\langle Q\rangle = \frac{d}{dt}\langle\Psi|\hat{Q}|\Psi\rangle$$

$$= \langle\frac{\partial\Psi}{\partial t}|\hat{Q}|\Psi\rangle + \langle\Psi|\frac{\partial\hat{Q}}{\partial t}|\Psi\rangle + \langle\Psi|\hat{Q}|\frac{\partial\Psi}{\partial t}\rangle$$

$$= \langle\frac{1}{i\hbar}\hat{H}\Psi|\hat{Q}|\Psi\rangle + \langle\Psi|\frac{\partial\hat{Q}}{\partial t}|\Psi\rangle + \langle\Psi|\hat{Q}|\frac{1}{i\hbar}\hat{H}\Psi\rangle$$

$$\langle\frac{dQ}{dt}\rangle = \langle\Psi|\frac{i}{\hbar}[\hat{H},\hat{Q}] + \frac{\partial\hat{Q}}{\partial t}|\Psi\rangle$$

even true without $\langle\Psi|\ldots|\Psi\rangle$ as an operator equation

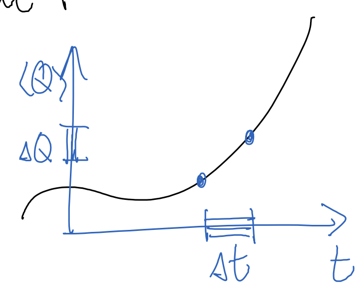
- if $[\hat{H},\hat{Q}]=0$ then $\hat{Q}$ is conserved ($\langle Q\rangle$ constant)

* how much time does it take $\langle Q\rangle$ to change?

$$\sigma_H\sigma_Q \geq \left|\frac{1}{2i}\langle[\hat{H},\hat{Q}]\rangle\right| = \left|\frac{1}{2i}\frac{\hbar}{i}\frac{d\langle Q\rangle}{dt}\right| = \frac{\hbar}{2}\left|\frac{d\langle Q\rangle}{dt}\right|$$

let $\Delta E = \sigma_H$, $\Delta t \equiv \frac{\sigma_Q}{|d\langle Q\rangle/dt|}$

time for $\langle Q\rangle$ to change σ_Q



then $\Delta E\,\Delta t \geq \frac{\hbar}{2}$.

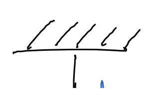
- a pure energy state never changes. $\sigma_H = 0 \Rightarrow \begin{matrix}\Delta Q = 0\\ \Delta t\to\infty\end{matrix}$

- sudden changes require infinite energy range $\Delta t\to 0 \Rightarrow \sigma_H\to\infty$

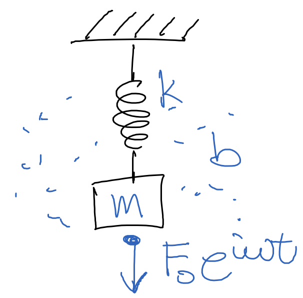
* Breit-Wigner resonances http://quantummechanics.ucsd.edu/ph130a/130_notes/node428.html

- damped, undriven classical harmonic oscillator

$kx + b\dot{x} + m\ddot{x} = 0$ let $x = e^{\alpha t}$



$$kx + b\dot{x} + m\ddot{x} = 0 \qquad \text{let } x = e^{\alpha t}$$

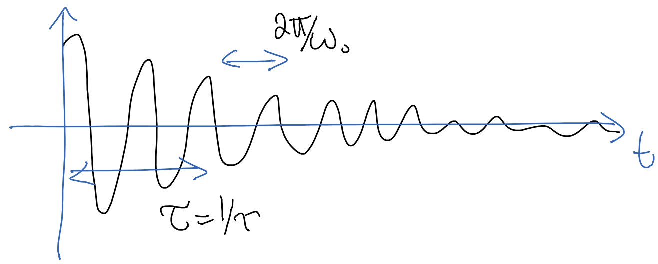


$$k + b\alpha + m\alpha^2 = 0 \qquad \alpha = \frac{-b}{2m} + \sqrt{\left(\frac{b}{2m}\right)^2 - \frac{k}{m}}$$

$$x = \left(A_1 e^{i\omega_0 t} + A_2 e^{-i\omega_0 t}\right) e^{-\Gamma/2\, t} \quad \text{where}$$

$\omega_0^2 = \Gamma^2 + k/m$ resonant frequency

$\Gamma \equiv \frac{b}{m} = 1/\tau$ lifetime

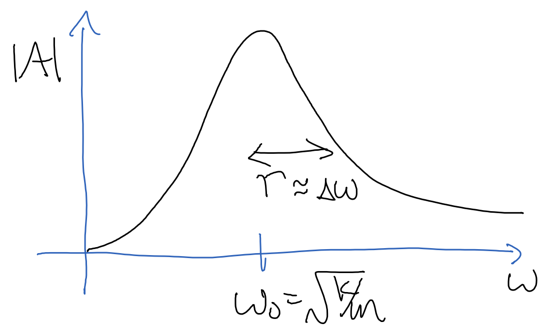


- drive it with an external frequency $F_{ext} = F_0 e^{i\omega t}$

$$F_{ext} - kx - b\dot{x} = m\ddot{x}$$

$$F_0 e^{i\omega t} = (-m\omega^2 + ib\omega + k) A e^{i\omega t}$$

$$A = \frac{F_0}{(k - m\omega^2) + ib\omega} \qquad |A|^2 = \frac{F_0^2}{(k - m\omega^2)^2 + b^2\omega^2}$$



- transition of energy E with lifetime $\tau = 1/\Gamma$

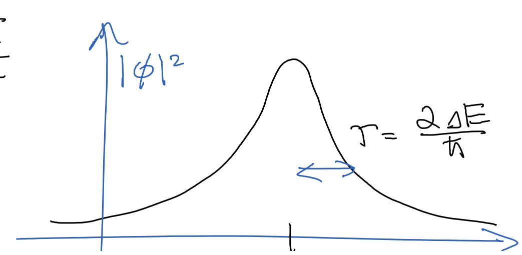
$$\Psi(x,t) = \psi(x) e^{-iEt/\hbar} e^{-\Gamma t/2} \quad \text{so that } P = \int dx |\psi|^2 \sim e^{-\Gamma t}$$

$$\phi(x,\omega) = \mathcal{F}\{\Psi(x,t)\} = \int dt\, e^{i\omega t}\left[\psi(x) e^{-iEt/\hbar} e^{-\Gamma t/2}\right]$$

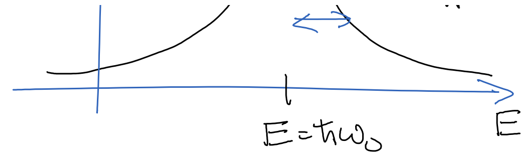
$$= \int_0^\infty dt\, e^{i(\omega - \omega_0 + \frac{i\Gamma}{2})} = \frac{i}{(\omega - \omega_0) + i\frac{\Gamma}{2}} \qquad E = \hbar\omega_0$$

thus $I(\omega) = \|\phi_i(x,\omega)\|^2 = \dfrac{1}{(\omega - \omega_0)^2 + \frac{\Gamma^2}{4}}$

"Breit-Wigner" line shape. FWHM Γ



Breit-Wigner line shape. FWHM



note: $\Delta E\cdot\Delta t = \hbar\frac{\Gamma}{2}\cdot\tau = \hbar/2$

- the longer a transition takes, the more oscillations go into the wave packet, and the better defined the frequency is: $\Delta t\,\Delta\omega \geq \frac{1}{2}$ (just like $\Delta x\cdot\Delta k \geq \frac{1}{2}$)