L35-3d Hilbert Space and Hamiltonians

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* Review solution of TDSE: (PDE)

$\hat{\mathcal{H}}\,\Psi(x,t) = \hat{E}\,\Psi(x,t)$ where $\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \hat{V}$, $\hat{E} = i\hbar\partial_t$

+ separation of variables: $\Psi(x,t) = \psi(x)\,\phi(t)$
  separate x on left, t on right, both equal constant E, leading to LHS, RHS eigenvalue equations

$$\hat{\mathcal{H}}\,\underbrace{\psi(x)} = E\,\underbrace{\psi(x)}_{\text{eigenfunction}} \qquad i\hbar\partial_t\,\underbrace{\phi(t)}_{\text{eigenfunction}} = E\,\underbrace{\phi(t)} = e^{-iEt/\hbar}$$

[TISE]

+ general solution is a linear combination of "tensor product" of building block eigenfunctions:

$$\Psi(x,t) = \sum_n c_n\,\psi_n(x)\,e^{-iE_nt/\hbar}$$ "skipping rope fns"

+ coefficients (components) determined from initial conditions $\Psi(x,0)$ and inner product.

$$\langle\psi_n|\Psi_0\rangle = \sum_{n'} c_{n'}\,\underbrace{\langle\psi_n|\psi_{n'}\rangle}_{\delta_{nn'}}\,\cancel{e^{-iE_n0/\hbar}}^{\,1} = c_n$$

$$|\Psi(x,t)\rangle = \underbrace{\sum_n |\psi_n\rangle\,e^{-iE_nt/\hbar}\langle\psi_n|}_{U(t)\,=\,e^{-i\hat{H}t/\hbar}}\,\Psi_0\rangle = U(0,t)\,|\Psi_0\rangle$$

* this same technique generalizes to 3-d:

+ 3-d wavefunctions: $\psi(\vec{r}) = \psi(x,y,z)$ or $\psi(r,\theta,\phi)$

norm: $\int d^3r\,|\psi(\vec{r})|^2 = 1 \qquad d^3r = dx\,dy\,dz = r^2\sin\theta\,d\theta\,d\phi\,dz.$

+ 3-d Schrödinger equation:

$V(x) \to V(x,y,z) = V(\vec{r})$ "Laplacian" ∇^2

$$\hat{T} = \frac{\hat{p}^2}{2m} \to \frac{\vec{p}^2}{2m} = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{p_z^2}{2m} = \frac{-\hbar^2}{2m}(\partial_x^2 + \partial_y^2 + \partial_z^2) = \frac{-\hbar^2 \nabla^2}{2m}$$

+ Separation of Variables: $\Psi(\vec{r},t) = \psi(\vec{r})\,\phi(t)$, $\psi(\vec{r}) = X(x)Y(y)Z(z)$

- use eigenfunctions to substitute each operator with its eigenvalue

* Example: 2-d infinite square well: $0<x<L$, $0<y<L$

$$\frac{-\hbar^2}{2m}\Big(\underbrace{\partial_x^2}_{-k_x^2} + \underbrace{\partial_y^2}_{-k_y^2}\Big) X(x)Y(y)\phi(t) = \underbrace{i\hbar\frac{\partial}{\partial t}}_{E=\hbar\omega} X(x)Y(y)\phi(t)$$

$$\partial_x^2 \sin(k_x x) = -k_x^2 \sin(k_x x) \text{ or } \cos(k_x x) \qquad \partial_t e^{i\omega t} = i\omega e^{i\omega t}$$

$$\partial_y^2 \sin(k_y y) = -k_y^2 \sin(k_y y) \text{ or } \cos(k_y y) \qquad \frac{\hbar^2}{2m}(k_x^2 + k_y^2) = \hbar\omega = E$$

- each spatial equation is a Sturm-Liouville system with boundary conditions
- apply B.C.'s separately in each dimension to get one quantum # per dimension: n, m for x, y.

+ Group activity: quantize k_x and k_y to obtain the energy spectrum E_{nm}
plot the node lines for each mode $X_n(x)Y_m(y)$

+ Summary & application: Wien's law $u(\nu) = \frac{8\pi\nu^2}{c^3} kT$

$$\#\text{modes} = d^3n = n^2\,dn\,d\Omega = \left(\frac{2L}{c}\right)^3 \nu^2 d\nu\, \frac{4\pi}{8}$$

density of states $g(\nu) = \frac{d^3 n}{d\nu \cdot L^3} = \frac{8\pi\nu^2}{c^3}$

spectral intensity $u(\nu) = g(\nu)\,\bar{\varepsilon} = \frac{8\pi\nu^2}{c^3} kT$