University of Kentucky, Physics 521
Homework #10, Rev. B, due Wednesday, 2016-02-10

0. Griffiths [2ed] Ch. 2 #17, Ch. 3 #35, #39, Ch. 4 #18, #19, #22.

1. Consider a two-dimensional isotropic harmonic oscillator with Hamiltonian $\mathcal{H}_{xy} = \frac{1}{2}\left(-\nabla^2 + \rho^2\right)$, where $\rho^2 = x^2 + y^2$, and $\hbar = m = \omega = 1$ for convenience (so that ρ is similar to the normalized coordinate ξ of Griffiths Eq. [2.71]).

a) Show that the Hamiltonian separates into two independent oscillators $\mathcal{H}_{xy} = \mathcal{H}_x + \mathcal{H}_y$ in cartesian coordinates, and thus the energy levels are $E_{n_x n_y} = n_x + n_y + 1$. Identify the degeneracy of each energy level, and write the wave functions of the lowest three levels, both in terms of Hermite polynomials $H_{n_x}(x)H_{n_y}(y)$ and in terms of creation operators $a_x^\dagger$ and $a_y^\dagger$ acting on the ground state $|n_x n_y\rangle = |00\rangle$, where $a_x = \frac{1}{\sqrt{2}}(x + ip_x)$ and $a_y = \frac{1}{\sqrt{2}}(y + ip_y)$ are the annihilation operators for each independent direction.

b) To obtain eigenstates of definite L_z, define annihilation operators for right and left circular quanta $a_r = \frac{1}{\sqrt{2}}(a_x - ia_y)$ and $a_\ell = \frac{1}{\sqrt{2}}(a_x + ia_y)$. Show that the only nonzero commutators between $a_r, a_r^\dagger, a_\ell, a_\ell^\dagger$ are $[a_r, a_r^\dagger] = [a_\ell, a_\ell^\dagger] = 1$. Show that $\mathcal{H}_{xy} = N_x + N_y + 1 = N_r + N_\ell + 1$ and $L_z = i(a_x a_y^\dagger - a_x^\dagger a_y) = N_r - N_\ell$, where $N_i = a_i^\dagger a_i$ as usual for $i = x, y, r, \ell$. Show also that $[\mathcal{H}_{xy}, a_i^\dagger] = a_i^\dagger$, so that a_x, a_y and a_r, a_ℓ act as ladder operators for two independent sets of quanta $|n_x n_y\rangle$ and $|n_r n_\ell\rangle$. Thus $E_{nm} = n + 1$, and $L_z = m$, where $n = n_r + n_\ell$ and $m = n_r - n_\ell$. Show that the allowed values of n are $2k + |m|$ where $k = 0, 1, 2 \ldots$ and $m = 0, \pm 1, \pm 2 \ldots$, and plot the energy levels E_{nm} versus m. Note the checker pattern with every other value of n missing.

c) Show that the Laplacian in cylindrical coordinates is $\nabla^2 = \frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial \rho^2}\sqrt{\rho} - \frac{(m-\frac{1}{2})(m+\frac{1}{2})}{\rho^2}$ using the eigenvalue $L_z \Phi_m(\phi) = m\Phi_m(\phi)$, where $\Phi_m(\phi) = e^{im\phi}$. Factor out the asymptotic dependence of ψ as $\rho \to 0$ and ∞ by making the substitution $\psi(\rho, \phi) = \rho^{|m|} e^{-\rho^2/2} F(\rho)\Phi_m(\phi)$ and use the energy eigenvalues $E_n = n + \frac{2}{2}$ to put the Schrödinger equation in the form

$$\frac{d^2 F}{d\rho^2} - \left(2\rho - \frac{2|m|+1}{\rho}\right)\frac{dF}{d\rho} + 2\left(n - |m|\right)F = 0. \tag{1}$$

Change variables to $f(u = \rho^2) = F(\rho)$ to obtain Laguerre's associated differential equation,

$$u\frac{d^2 f}{du^2} + \left(|m| + 1 - u\right)\frac{df}{du} + kf = 0, \tag{2}$$

with the solution $f(u) = L_k^{|m|}(u)$, where $k = \frac{1}{2}(n - |m|) = 0, 1, 2 \ldots$. Write out the lowest three energy levels and show that the ground state is equal to that in part a). Write the next two degenerate levels as linear combinations of ψ_{10} and ψ_{01} from part a).

d) [bonus] Show that the three-dimensional Laplacian has a similar form after factoring out angular momentum: $\nabla^2 = \frac{1}{r}\frac{\partial^2}{\partial r^2}r - \frac{(l)(l+1)}{r^2}$, and by analogy with part c) solve for the normalized energy eigenfunctions

$$\psi_{nlm} = (-1)^k \sqrt{\frac{2k!}{\Gamma(n-k+\frac{3}{2})}} r^l e^{-\frac{1}{2}r^2} L_k^{l+\frac{1}{2}}(r^2) Y_{lm}(\theta,\phi) \tag{3}$$

and eigenvalues $E_{nlm} = n + \frac{3}{2}$, where $n = 2k + l$ with $k = 0, 1, 2 \ldots$. Determine the degeneracy of each energy level and make a correspondence of states with linear combinations of the cartesian solution $E_{n_x n_y n_z} = n_x + n_y + n_z + \frac{3}{2}$.

e) [bonus] Show that the one-dimensional Laplacian can also be put into the same form, $\nabla^2 = \frac{\partial^2}{\partial \xi^2} - \frac{(p-1)p}{\xi^2}$, using the 'radial' coordinate $\xi = |x|$ and discrete 'angle' $\sigma = \mathrm{sgn}(x) = x/|x|$, and factoring out the *parity harmonics* $\Pi_p(\sigma) = e^{i\pi p\sigma} = (-1)^{\sigma p}$. (Note the second term, the centrifugal barrier, is always zero!) Show that $p = 0, 1$ are the eigenvalues for even, and odd functions, respectively. By analogy with parts c) and d), show that $E_{np} = n + \frac{1}{2}$, where $n = 2k + p$, and write the 1-d eigenfunctions in terms of Laguerre polynomials. Equate these with the normal Hermite solutions to obtain the two relations [Abramowitz & Stegun, Eqs. 22.5.40-41].

$$\begin{aligned} H_{2m}(x) &= (-1)^m 2^{2m} m! L_m^{-\frac{1}{2}}(x^2) && (4)\\ H_{2m+1}(x) &= (-1)^m 2^{2m+1} m! x L_m^{\frac{1}{2}}(x^2). && (5) \end{aligned}$$

f) [bonus] Show that the same dimensional symmetry of solutions occurs for the free particle wave equation:

$$\begin{aligned} \Psi(\xi,\sigma) &= \sqrt{\xi} J_{p-\frac{1}{2}}(k\xi)\, \Pi_p(\sigma) && (6)\\ \Psi(\rho,\phi) &= J_m(k\rho)\Phi_m(\phi) && (7)\\ \Psi(r,\theta,\phi) &= \frac{1}{\sqrt{r}} J_{l+\frac{1}{2}}(kr) Y_{lm}(\theta,\phi) && (8) \end{aligned}$$

The order of the Bessel function is raised by a half as you go to a higher dimension. The one- and three-dimensional solutions are usually written using circular functions e^{ikx} and spherical Bessel functions $j_l(kr)$.