L51-Disk Square Well

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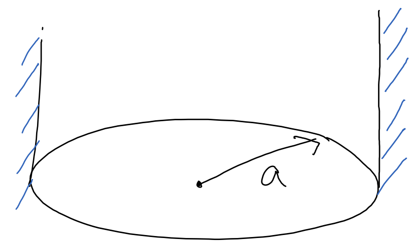
* See Ex 4.1 Griffiths for infinite spherical well

Potential: $V(r) = \begin{cases} 0 & r<a \\ \infty & r>a \end{cases}$ $\qquad E_{n\ell} = \frac{\hbar^2 k_{n\ell}}{2m}$

Solution $\psi(\vec{r}) = \sum_{n\ell m} c_{n\ell}\, j_\ell(k_{n\ell} r) Y_{\ell m}(\theta,\phi)$ where $j_\ell(\underbrace{k_{n\ell}a}_{\beta_{n\ell}}) \equiv 0$

* 2-d case: infinite spherical well on a disk

Potential: $V(\rho,\phi) = \begin{cases} 0 & \rho<a \\ \infty & \rho>a \end{cases}$



$$\hat{H} = \frac{-\hbar^2 \nabla^2}{2m} + \hat{V} = \frac{-\hbar^2}{2m}\left(\frac{1}{\rho}\underbrace{\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho}}_{①} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)$$

$$= \frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right). \qquad = \frac{-\hbar^2}{2m}\left(\underbrace{\frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial\rho^2}\sqrt{\rho}}_{②} + \frac{1}{4\rho^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right)$$

① $\frac{\partial}{\partial\rho}(\rho f') = f' + \rho f''$

so $\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}$

② $\frac{\partial^2}{\partial\rho^2}(\rho^{1/2} f) = \frac{\partial}{\partial\rho}\left(\frac{1}{2}\rho^{-1/2} f + \rho^{1/2} f'\right)$

$= \frac{1}{2}\cdot\frac{-1}{2}\rho^{-3/2} f + \frac{1}{2}\rho^{-1/2} f' + \frac{1}{2}\rho^{-1/2} f' + \rho^{1/2} f''$

so $\frac{1}{\sqrt{\rho}}\frac{\partial^2}{\partial\rho^2}\sqrt{\rho} + \frac{1}{4\rho^2} = \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho}$ also!

note the pattern: $\nabla_d^2 = \left(\frac{1}{r}\right)^{\frac{d+1}{2}}\frac{\partial^2}{\partial r^2}(r)^{\frac{d+1}{2}} + ?\cdot\frac{1}{r^2}$ in d-dimensions.

Azimuthal eigenfunctions: $\frac{\partial^2}{\partial\phi^2} e^{im\phi} = -m^2 e^{im\phi}$

Radial Equation: $\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2}\right) R(r) = E_{nm}\cdot R(r) = \frac{\hbar^2 k^2}{2m}\cdot R(r)$

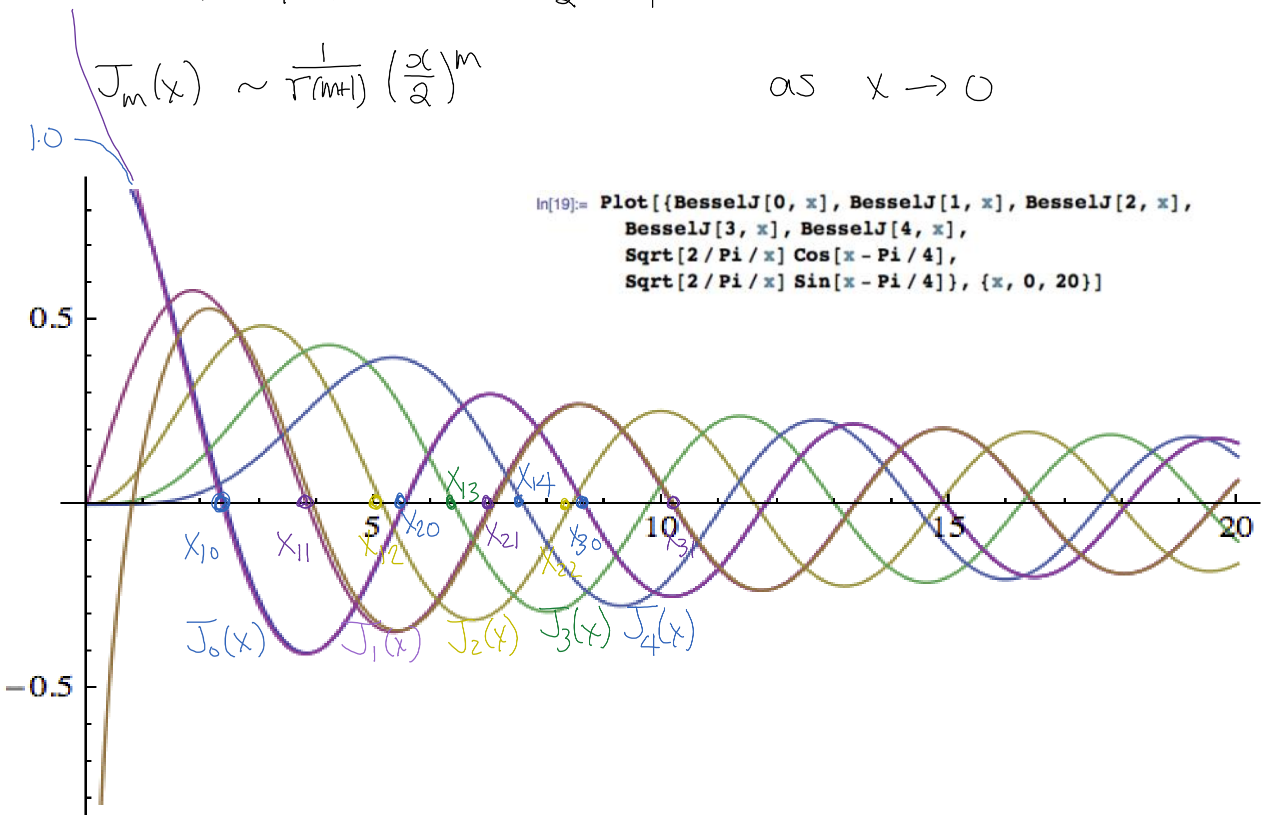
$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{m^2}{\rho^2}\right) R(r) = -k^2 R(r)$ $\qquad$ let $x = k\rho$ (dimensionless)

$\left(x^2\frac{d^2}{dx^2} + x\frac{d}{dx} + x^2 - m^2\right) J_m(k\rho) = 0$ $\qquad$ Bessel's equation.

The solutions are Bessel functions of order m.
"sine waves spreading out over a circle"

$$J_m(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \quad \text{as} \quad x \to \infty$$

$$J_m(x) \sim \frac{1}{\Gamma(m+1)} \left(\frac{x}{2}\right)^m \quad \text{as} \quad x \to 0$$



The second independent solution $N_m(x)$ blows up as $x \to 0$

B.C. #1: $\psi(\rho, \phi)$ continuous as $\rho \to 0$: use $J_m(x)$ not $N_m(x)$

$$\psi(\rho,\phi) = J_m(k\rho)\, e^{im\phi} \sim \rho^m e^{im\phi} \sim (\rho e^{i\phi})^m \sim (x+iy)^m$$

as $\rho \to 0$ is a smooth function of x, y.

note: $V(\rho, \phi) = \rho^m e^{im\phi}$ is the solution of Laplace eq'n: $k \to 0$

B.C. #2: $\psi(a, \phi) = 0$ because $\psi(\rho, \phi) = 0$ if $\rho > a$

$J_m(ka) = 0$ let $k_{nm} a = x_{nm}$, "the n^{th} zero of $J_m(x)$"

The general solution is:

$$\psi_{nm}(\rho, \phi) = \sum_{n,m} c_{nm} J_m(k_{nm}\rho)\, e^{im\phi}$$

The energy spectrum is: $E_{nm} = \frac{\hbar^2 k_{nm}}{2m} = \frac{\hbar^2}{2ma^2} (x_{nm})^2$

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Each level $\pm m$ is doubly degenerate except $m=0$. This corresponds to cylindrical symmetry.