L52-Angular Momentum-Eigenvalues

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* we've labeled the angular part of $\hat{T}$ "angular momentum"

$$\frac{\hat{p}^2}{2m} = \frac{-\hbar^2\nabla^2}{2m} = \frac{-\hbar^2}{2m}\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{\hbar^2}{2mr^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) = \hat{T} + \frac{\hat{L}^2}{2I}$$

$I = mr^2$

$$\hat{L}^2 = -\hbar^2\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) \qquad \hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

based on analogy of angular kinetic energy.

* definition: $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}} = [y p_z - z p_y,\ z p_x - x p_z,\ x p_y - y p_x]$

example: $L_z = -i\hbar\frac{\partial}{\partial\phi} = -i\hbar\left(\frac{\partial x}{\partial\phi}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\phi}\frac{\partial}{\partial y}\right) = -y p_x + x p_y$

$x = \rho\cos\phi \quad \frac{\partial x}{\partial\phi} = -y \qquad y = \sin\phi \quad \frac{\partial y}{\partial\phi} = x$

* commutation relations: cannonical relations: $\boxed{[r_i, p_j] = i\hbar\,\delta_{ij}}$

$$[L_x, L_y] = [y p_z - z p_y,\ z p_x - x p_z]$$

$$= [y p_z, z p_x] + \cancel{[y p_z, -x p_z]} + \cancel{[-z p_y, z p_x]} + [-z p_y, -x p_z]$$

$$= y[p_z, z]p_x + x[z, p_z]p_y = i\hbar(-y p_x + x p_y) = i\hbar L_z$$

(using the identities: $[A, BC] = A(BC) - (BC)A = B(AC - CA) + (AB - BA)C = B[A,C] + [A,B]C$

$[XY, Z] = -[Z, XY] = -X[Z,Y] - [Z,X]Y = [X,Z]Y - X[Y,Z]$)

by cyclic permutation, $\boxed{[L_i, L_j] = \varepsilon_{ijk}\, i\hbar L_k}$

or formally, $\boxed{\hat{L} \times \hat{L} = i\hbar \hat{L}}$

this forms a generalized definition of angular momentum!

* define $\hat{L}^2 \equiv L_x^2 + L_y^2 + L_z^2$

then $[L_x, L^2] = \cancel{[L_x, L_x^2]}^{\,0} + [L_x, L_y^2] + [L_x, L_z]^2$

$= [L_x, L_y]L_y + L_y[L_x, L_y] + [L_x, L_z]L_z + L_z[L_x, L_z]$

$= i\hbar(L_zL_y + L_yL_z) - i\hbar(L_yL_z + L_zL_y) = 0$

thus $\boxed{[L^2, \vec{L}] = 0}$

we can choose L^2, L_z to be a complete set of commuting operators over the space of angular functions. (these appear in ∇^2!)

* factor $L^2 = L_x^2 + L_y^2 + L_z^2 \overset{?}{=} (L_x + iL_y)(L_x - iL_y) + L_z^2$ NO! commutation relations!

define: $L_\pm \equiv L_x \pm iL_y \qquad L_\pm^\dagger = L_\mp$ ladder operators $[L_\pm, L^2] = 0$ still.

$L_+L_- = (L_x + iL_y)(L_x - iL_y) = L_x^2 + L_y^2 - i[L_x, L_y] = L_x^2 + L_y^2 + \hbar L_z$

$L_-L_+ = L_x^2 + L_y^2 - \hbar L_z$ so $[L_+, L_-] = 2[L_x, L_y] = 2\hbar L_z$

$[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y] = i\hbar(L_y \mp iL_x] = \pm\hbar L_\pm$

thus $L_z L_\pm = (L_\pm \pm \hbar) L_z$ it raises or lowers eigenvalues

* Ladder property: let $|lm\rangle$ be an eigenstate of L^2, L_z

where $\hat{L}^2|lm\rangle = \lambda|lm\rangle \qquad L_z|lm\rangle = \mu|lm\rangle$

consider the new state $L_\pm|lm\rangle$

$L^2(L_\pm|lm\rangle) = L_\pm L^2|lm\rangle = L_\pm \lambda|lm\rangle = \lambda(L_\pm|lm\rangle)$

$L_z(L_\pm|lm\rangle) = (L_\pm \pm \hbar)L_z|lm\rangle = (\mu \pm \hbar)(L_z|lm\rangle)$

* let the maximum eigenvalue of L_z be $\mu = \hbar l$ so $L_+|ll\rangle = 0$

then $L^2|ll\rangle = [(L_-L_+ + \hbar L_z) + L_z^2]|ll\rangle = \hbar^2 l(l+1)|ll\rangle$

* let the maximum eigenvalue of L_z be $\mu = \hbar \ell$ so $L_+ |\ell\ell\rangle = 0$

then $L^2 |\ell\ell\rangle = \left[\left(L_- \cancel{L_+} + \hbar L_z \right) + L_z^2 \right] |\ell\ell\rangle = \underbrace{\hbar^2 \ell(\ell+1)}_{\lambda} |\ell\ell\rangle$

at the bottom rung, $L_- |\ell\bar{\ell}\rangle = 0$

then $L^2 |\ell\bar{\ell}\rangle = \left[\left(L_+ \cancel{L_-} - \hbar L_z \right) + L_z^2 \right] |\ell\bar{\ell}\rangle = \underbrace{\hbar^2 \bar{\ell}(\bar{\ell}-1)}_{\text{also } \lambda} |\ell\bar{\ell}\rangle$

thus $\ell(\ell+1) = \bar{\ell}(\bar{\ell}-1)$: $\bar{\ell} = \ell+1$ or $\bar{\ell} = -\ell$ but $\bar{\ell} < \ell$

so $-\hbar\ell \leq \mu \leq \hbar\ell$ in steps of $\hbar$, which implies

$\ell \in \{0, \frac{1}{2}, 1, 1\frac{1}{2}, 2, \dots\}$; $\mu = \hbar m$ $m \in \{-\ell, -\ell+1, \dots \ell-1, \ell\}$

that is why we index the state by [half] integers ℓ, m.

$$\boxed{\begin{aligned} L^2 |\ell m\rangle &= \hbar^2 \ell(\ell+1) |\ell m\rangle \\ L_z |\ell m\rangle &= \hbar m \ |\ell m\rangle \end{aligned}}$$

In our case $|\ell m\rangle \sim Y_{\ell m}(\theta, \phi)$

$\ell = 0, 1, 2, \dots$ for orbital angular momentum