Properties of Commutators

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(3)

* Multilinear

$$[\alpha_i A_i, B] = \alpha_i [A_i, B]$$
$$[A, \beta_i B_i] = \beta_i [A, B_i]$$

* Antisymmetric

$$[A,B] = -[B,A] \Rightarrow [A,A] = 0$$

* Jacobian identity.

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

Lie Algebra

like a cross product.

* Derivation - like a derivative: $d(BC) = dB \cdot C + B \cdot dC$

$$[A, BC] = [A,B]C + B[A,C]$$
$$[AB, C] = [A,C]B + A[B,C]$$

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* Hadamard Lemma (Campbell-Baker-Hausdorff expansion)

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots$$
$$\equiv e^{ad(A)} \cdot B \qquad \text{where} \qquad ad(A) \cdot B = [A,B]$$

$$\ln(e^A e^B) = A + B + \frac{1}{2}[A,B] + \frac{1}{12}\left([A,[A,B]] + [B,[B,A]]\right) + \dots \equiv C$$

- do a Taylor expansion of $e^A = 1 + A + \frac{1}{2!}A^2 + \dots$, e^B, and e^C
- follows from above properties.