L54-Spin

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* QM spin is NOT: Classical Spin !

- it is not "internal" angular momentum (of constituents inside the particle)

$$\vec{J} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i (\vec{r}_{cm} + \vec{r}_i{}') \times (\vec{p}_{i\,cm} + \vec{p}_i{}')$$

$$\sum_i m_i (1, \vec{r}_i) \equiv m(1, \vec{r}_{cm})$$

$$\vec{p}_{cm} \equiv \sum_i \vec{p}_i = \sum_i \vec{p}_{i\,cm}$$

$$= \underbrace{\sum_i \vec{r}_{cm} \times \vec{p}_{i\,cm}}_{} + \cancel{\vec{r}_{cm} \times \vec{p}_i{}'} + \cancel{\vec{r}_i{}' \times m_i \dot{\vec{r}}_{cm}} + \underbrace{\vec{r}_i{}' \times \vec{p}_i{}'}_{}$$

$$= \vec{L} = \vec{r}_{cm} \times \vec{p}_{cm} \quad + \quad \vec{S} = \sum_i \vec{r}_i{}' \times \vec{p}_i{}'$$

- we will learn QM "addition of angular momenta" later
- occurs even for point-like ("fundamental") particles !
- has the wrong tensorial nature: spin ½ not scalar or vector !
- even spin-1 (which IS a vector) looks different !
- not neccesarily QM - also classical ! (think polarization)

* QM spin IS:

- intrinsic angular momentum (invariant; nonlocal - independent of $\vec{r}$)
  Ohnian, A.J.P 54 500 (1986), Gsponer, arXiv: physics/0308627 (2003)
- a property of waves ! : it is the "angular momentum/magnetic moment generated by circulation of energy/current in the wave field
  Belinfante, Physica 6 887 (1939), Gordon, Z.Phys 50 630 (1928), Humblet Physica 10, 585 (1943)
- it is caused/characterized by the tensorial nature of the field: a scalar has no spin $S=0$, a vector has $S=1$, Quadrupole/metric tensor $S=2$

eg#1 $\vec{L}$ vector $\rightarrow L_-, L_z, L_+$ ($m=-1,0,1$) "spherical tensor components"

eg#2 photon: $\vec{E} = (\hat{x} E_x + \hat{y} E_y) e^{ikz-wt}$ has 2 polarizations: $\varepsilon_x = \hat{x}$, $\varepsilon_y = \hat{y}$
circular polarization: $\varepsilon_+ = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y})$ and $\varepsilon_- = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y})$
has $S=1$, $M_S = \pm 1$ — note $M_S = 0$ (longitudinal pol) is missing
note: ang. momentum $\dfrac{\int d\vec{S} \equiv \vec{E} \times \vec{A}\, dV}{\int dU = \frac{1}{2}(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H})} = \dfrac{1}{\omega}$ $\quad U = \hbar\omega$; $S = 1 \cdot \hbar$ quantized for photons.

eg#3 $\vec{r}_1 \otimes \vec{r}_2 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix} (x_2\ y_2\ z_2) = \begin{pmatrix} x_1x_2 & x_1y_2 & x_1z_2 \\ y_1x_2 & y_1y_2 & y_1z_2 \\ z_1x_2 & z_1y_2 & z_1z_2 \end{pmatrix} \sim \underbrace{\vec{r}_1 \cdot \vec{r}_2}_{\text{scalar } S=0} + \underbrace{\vec{r}_1 \times \vec{r}_2}_{\text{pseudovector } S=1} + \underbrace{\tfrac{3}{2}(\vec{r}_1\vec{r}_2 + \vec{r}_2\vec{r}_1 - \vec{r}_1 \cdot \vec{r}_2 \, I)}_{\text{symmetric traceless tensor } S=2}$

- so spin itself is the $\nearrow$ of the fields (multiple components/polarizations) is classical, has geometric interpretation, and can be quantized.

- perhaps the real mystery is that $S = \frac{1}{2}, \frac{3}{2}, \ldots$ exist in nature, as predicted by the operator analysis of angular momentum.

<u>BOSONS</u> $S=0$: Higgs H, scalar mesons π, η, K — $S=1$: vector bosons/mesons $\gamma, W^\pm, Z, g, \rho, \omega, \ldots$ — $S=2$: graviton $g_{\mu\nu}$

<u>FERMIONS</u> $S=\frac{1}{2}$: quarks & leptons $\begin{matrix} u & c & t \\ d & s & b \end{matrix}$ $\begin{matrix} e & \mu & \tau \\ \nu_e & \nu_\mu & \nu_\tau \end{matrix}$, nucleons (octet) $p, n, \Sigma, \Lambda, \Xi$ — $S=3/2$: (decuplet) $\Delta, \Sigma, \Xi, \Omega$

- so the question isn't "what is spin", but rather "what is the geometrical interpretation of a 2-component field?"
- convention answer: "an irreducible representation of the rotation group" ie. it just mathematically works out that way. (from operator analysis)

* Spin-1/2 system: 1) $\vec{S} \times \vec{S} = i\hbar \vec{S}$ [angular momentum]
Postulates 2) s is an intrinsic property of the particle

- for $s = \frac{1}{2}$ we can't write a wave function, we don't want to anyway.
- the state is: $|sm\rangle$ defined as the simultaneous eigenstate of:

$$S^2 |sm\rangle = \hbar^2 s(s+1) \, |sm\rangle \quad \rightarrow \quad 3/4 \, \hbar^2 \quad (s = 1/2)$$
$$S_z |sm\rangle = \hbar m \, |sm\rangle \qquad m = \pm 1/2$$

$$S_\pm |sm\rangle = \hbar \sqrt{s(s+1) - m(m\pm1)} \, |s, m\pm1\rangle \quad \rightarrow \quad \begin{cases} 0 & \text{off the edge} \\ \hbar & \text{raising/lowering} \end{cases}$$

used to get $S_x \pm i S_y = S_\pm$

* cannonical matrix representation of $s = 1/2$:

ie $\vec{V} = V_x \hat{x} + V_y \hat{y}$

$$\chi = a \chi_+ + b \chi_- \qquad \chi_+ \equiv |\tfrac{1}{2}\ \tfrac{+1}{2}\rangle \qquad \chi_- \equiv |\tfrac{1}{2}\ \tfrac{-1}{2}\rangle$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$S^2 \begin{pmatrix} a \\ b \end{pmatrix} = \tfrac{3}{4}\hbar \begin{pmatrix} a \\ b \end{pmatrix} \qquad \text{so } S^2 = \tfrac{3}{4}\hbar I$$

PAULI MATRICES

$$S_z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +\tfrac{1}{2}\hbar \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad S_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\tfrac{1}{2}\hbar \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \text{so } S_z = \tfrac{1}{2}\hbar \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$S_+ \left(\begin{matrix} 1 \\ 0 \end{matrix} \middle| \begin{matrix} 0 \\ 1 \end{matrix} \right) = \hbar \left(\begin{matrix} 0 \\ 0 \end{matrix} \middle| \begin{matrix} 1 \\ 0 \end{matrix} \right) \qquad \text{so } S_+ = \hbar \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$S_- \left(\begin{matrix} 1 \\ 0 \end{matrix} \middle| \begin{matrix} 0 \\ 1 \end{matrix} \right) = \hbar \left(\begin{matrix} 0 \\ 1 \end{matrix} \middle| \begin{matrix} 1 \\ 0 \end{matrix} \right) \qquad \text{so } S_- = \hbar \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$S_x = \tfrac{1}{2}(S_+ + S_-) = \tfrac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$S_y = \tfrac{1}{2i}(S_+ - S_-) = \tfrac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

check: $S_x^2 + S_y^2 + S_z^2 = S_\pm S_\mp \pm \hbar S_z + S_z^2 = S^2 = \tfrac{3}{4}\hbar^2 I$

* Pauli algebra — a "Clifford algebra".

- let $\vec{\sigma} \equiv [\sigma_x, \sigma_y, \sigma_z]$ then $\sigma_i \sigma_j = \underbrace{\delta_{ij} I}_{\text{DOT (sym)}} + \underbrace{i\,\varepsilon_{ijk}\sigma_k}_{\text{CROSS. (antisym)}}$

  "basis frame"! $\quad (\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B}\, I + i\,\vec{\sigma}\cdot\vec{A}\times\vec{B}$

- the fact that $\vec{\sigma}\times\vec{\sigma} = 2i\vec{\sigma}$ mimics the cross product allows $\vec{\sigma}$ to "generate" rotations.

- decomposition of $GL_2(\mathbb{C})$ 2x2 complex matrices:

$$M = \underbrace{I a}_{\text{Scalar}} + \underbrace{\vec{\sigma}\cdot\vec{b}}_{\text{Vector}} + \underbrace{i\vec{\sigma}\cdot\vec{c}}_{\text{Axial vector}} + \underbrace{i I d}_{\text{Pseudoscalar}}$$

* eigenvectors of σ_x or σ_y

$$\sigma_x \chi_\pm^{(x)} = \begin{pmatrix} 0 & 1 \\ \cdot & \end{pmatrix} \begin{pmatrix} a \\ \cdot \end{pmatrix} = \lambda \begin{pmatrix} a \\ \cdot \end{pmatrix} \qquad \begin{vmatrix} 0-\lambda & 1 \\ \cdot & \end{vmatrix} = \lambda^2 - 1 = 0 \quad \lambda = \pm 1$$

$$\sigma_x \chi_\pm^{(x)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \qquad \begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \quad \lambda = \pm 1$$

$$\lambda = 1: \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \lambda = -1: \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$U^{(x)} = \left(\chi_+^{(x)} \; \chi_-^{(x)} \right) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \boxed{\sigma_x U^{(x)} = U^{(x)} \underbrace{\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}_{\sigma_z}}$$

* what is the state of an electron?

$$|n l m_l m_s\rangle = |n l m_l\rangle \otimes |s m_s\rangle \cong \phi_{nlm_l}(r,\theta,\phi) \otimes \chi_{m_s} = \begin{pmatrix} \psi_{nlm}^{+} \\ \psi_{nlm}^{-} \end{pmatrix}(r,\theta,\phi)$$

$$|\psi\rangle = \sum_{m_s = \pm 1/2} \sum_{nlm} \phi_{nlm}^{m_s}(r,\theta,\phi)\, \chi_{m_s}$$

- It is a "spinor field" – 2-component wave function with certain transformation properties.

- What are these transformation properties?
- What is the geometrical significance of them?