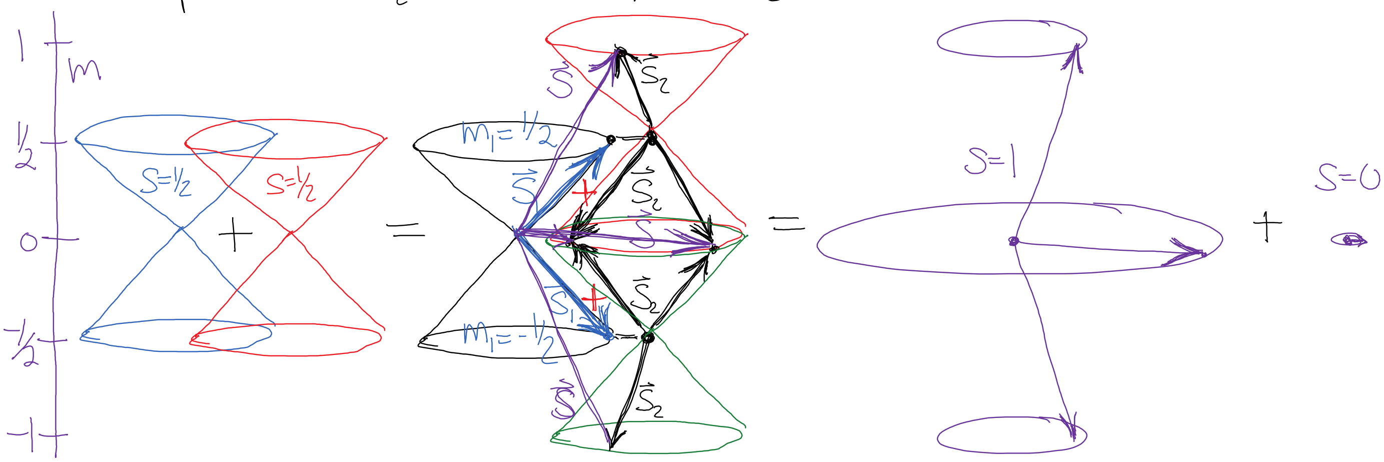
L56-Addition of Angular Momenta

Friday, February 5, 2016 08:17

* classically $\vec{L} = \vec{L}_1 + \vec{L}_2$
  just adds component by component

* quantum mechanically, only one component is observable.
  however the operators still add: $\vec{J} = \vec{J}_1 + \vec{J}_2$

  - example: $\vec{S}_1 + \vec{S}_2$ where $s_1 = 1/2$ $s_2 = 1/2$



* what do the states look like?
  it is a "tensor product" of the two spaces:
  - all possible combinations of basis states

$s_1=\frac{1}{2}$ \\ $s_2=\frac{1}{2}$	↑ m_2 $+\frac{1}{2}$	↓ $-\frac{1}{2}$
↑ $+\frac{1}{2}$ m_1	↑↑ m=1	↑↓ m=0
↓ $-\frac{1}{2}$	↓↑	↓↓ m=-1

"addition" of spin produces
② $\vec{S}_1$ states × ② $\vec{S}_2$ states
= ④ "product" states:

$$|s_1 m_1\rangle \otimes |s_2 m_2\rangle = |m_1 m_2\rangle \in \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$$

note $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ are different states!

* how do you operate on these states?

  $\vec{S}_1$ acts on $|s_1 m_1\rangle$ and ignores $|s_2 m_2\rangle$ (like a constant in front of the operator)

  $\vec{S}_2$ ignores $|s_1 m_1\rangle$ and acts on $|s_2 m_2\rangle$

$$(S_z = S_{1z} + S_{2z})\,\underbrace{|s_1 m_1\rangle |s_2 m_2\rangle} = \underbrace{\left(S_{1z}|s_1 m_1\rangle\right)}|s_2 m_2\rangle + |s_1 m_1\rangle \underbrace{\left(S_{2z}|s_2 m_2\rangle\right)}$$

$$(S_z = S_{1z} + S_{2z})\underbrace{|s_1 m_1\rangle |s_2 m_2\rangle}_{\text{these do not act on eachother}} = \underbrace{(S_{1z}|s_1 m_1\rangle)}_{\hbar m_1 |s_1 m_1\rangle}|s_2 m_2\rangle + |s_1 m_1\rangle \underbrace{(S_{2z}|s_2 m_2\rangle)}_{\hbar m_2 |s_2 m_2\rangle}$$

$$m = m_1 + m_2 \qquad = \hbar \underbrace{(m_1 + m_2)}_{m} |s_1 m_1\rangle |s_2 m_2\rangle$$

$$S_z|\uparrow\uparrow\rangle = \hbar \underbrace{\left(\tfrac{1}{2}+\tfrac{1}{2}\right)}_{m=1} |\uparrow\uparrow\rangle \qquad S_z|\uparrow\downarrow\rangle = \hbar \underbrace{\left(\tfrac{1}{2}-\tfrac{1}{2}\right)}_{m=0} |\uparrow\downarrow\rangle$$

$$S_z|\downarrow\uparrow\rangle = \hbar \underbrace{\left(-\tfrac{1}{2}+\tfrac{1}{2}\right)}_{m=0} |\downarrow\uparrow\rangle \qquad S_z|\downarrow\downarrow\rangle = \hbar \underbrace{\left(-\tfrac{1}{2}-\tfrac{1}{2}\right)}_{m=-1} |\downarrow\downarrow\rangle$$

* can we form multiplets out of these states with $m = -s, \dots s-1, s$?

$m = 1, \boxed{0}, -1$ corresponds to $s=1$ "triplet"

$m = \boxed{0}$ corresponds to $s=0$ "singlet"

which state goes where ?

– note that $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ are both symmetric w/r particle exchange $1 \leftrightarrow 2$ (more about this next chapter)

the linear combination $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$ is also symmetric

thus the triplet is $\left\{ \uparrow\uparrow, \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), \downarrow\downarrow \right\}$

the singlet is antisymmetric: $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$ has $s=0$

* confirmation 1: try ladder operators: $S_\pm = S_{1\pm} + S_{2\pm}$

$$S_- \uparrow\uparrow = (S_-\uparrow)\uparrow + \uparrow(S_-\uparrow) \propto \downarrow\uparrow + \uparrow\downarrow$$

$$S_-\left([\uparrow\downarrow] \pm [\downarrow\uparrow]\right) \propto \left[(S_-\uparrow)\downarrow + \uparrow\cancel{(S_-\downarrow)}\right] \pm \left[\cancel{(S_-\downarrow)}\uparrow + \downarrow(S_-\uparrow)\right] \propto \downarrow\downarrow \pm \downarrow\downarrow \propto \begin{matrix}\downarrow\downarrow \\ 0\end{matrix}$$

* confirmation 2: calculate $S^2 = (\vec{S}_1 + \vec{S}_2)^2 = \underbrace{S_1^2}_{\hbar^2 \frac{1}{2}\left(\frac{3}{2}\right)} + 2\vec{S}_1 \cdot \vec{S}_2 + \underbrace{S_2^2}_{\hbar^2 \frac{1}{2}\left(\frac{3}{2}\right)}$

$$S_1 \cdot S_2 = \frac{\hbar^2}{4} \vec{\sigma}_1 \cdot \vec{\sigma}_2 = \frac{\hbar^2}{4}(\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z})$$

$$\sigma_{1x}\sigma_{2x}|\uparrow\uparrow\rangle = (\sigma_{1x}\uparrow)(\sigma_{2x}\uparrow) = \downarrow \cdot \downarrow = |\downarrow\downarrow\rangle$$

$$\sigma_{1y}\sigma_{2y}|\uparrow\uparrow\rangle = (\sigma_{1y}\uparrow)(\sigma_{2y}\uparrow) = i\downarrow \cdot i\downarrow = -|\downarrow\downarrow\rangle$$

these cancel.

$$\sigma_{1y}\sigma_{2y}|\uparrow\uparrow\rangle = (\sigma_{1y}\uparrow)(\sigma_{2y}\uparrow) = i\downarrow\cdot i\downarrow = -|\downarrow\downarrow\rangle \quad \Big\} \text{ cancel.}$$

$$\sigma_{1z}\sigma_{2z}|\uparrow\uparrow\rangle = (\sigma_{1z}\uparrow)(\sigma_{2z}\uparrow) = \uparrow\cdot\uparrow = |\uparrow\uparrow\rangle$$

thus $S^2|\uparrow\uparrow\rangle = \hbar^2\left(\frac{1}{2}\cdot\frac{3}{2} + 2\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{3}{2}\right)|\uparrow\uparrow\rangle = \hbar^2\underbrace{(1)}_{S=1}(2)|\uparrow\uparrow\rangle$

others: $\vec{\sigma}\cdot\vec{\sigma}\,|\uparrow\downarrow\rangle = \downarrow\uparrow + i(-i)\downarrow\uparrow + 1\cdot(-1)\uparrow\downarrow = 2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle$

$$\vec{\sigma}\cdot\vec{\sigma}\,|\downarrow\uparrow\rangle = \uparrow\downarrow + (-i)i\uparrow\downarrow + -1\cdot1\uparrow\downarrow = 2|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

$$\vec{\sigma}\cdot\vec{\sigma}\,|\downarrow\downarrow\rangle = \uparrow\uparrow + (-i)(-i)\uparrow\uparrow + -1\cdot-1\downarrow\downarrow = |\downarrow\downarrow\rangle$$

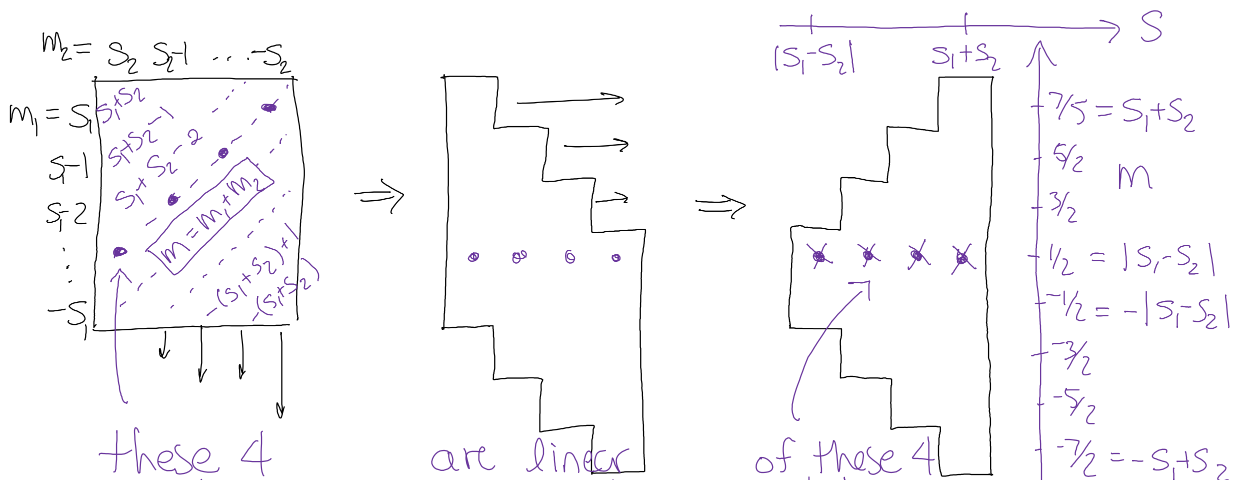
thus
$$S^2|\uparrow\uparrow\rangle = \hbar^2\cdot2|\uparrow\uparrow\rangle$$
$$S^2|\uparrow\downarrow\rangle = \hbar^2\cdot(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$
$$S^2|\downarrow\uparrow\rangle = \hbar^2\cdot(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$
$$S^2|\downarrow\downarrow\rangle = \hbar^2\cdot2|\downarrow\downarrow\rangle$$

$$S = \begin{array}{c|cccc|c} & \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow & \\ \hline \uparrow\uparrow & 2 & & & & m=1 \\ \uparrow\downarrow & & 1 & 1 & & m=0 \\ \downarrow\uparrow & & 1 & 1 & & \\ \downarrow\downarrow & & & & 2 & m=-1 \\ \hline & m=1 & m=0 & & m=-1 & \end{array}$$

diagonalize centre block:

$$\begin{pmatrix}1&1\\1&1\end{pmatrix}\underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{\frac{1}{\sqrt{2}}(\uparrow\downarrow+\downarrow\uparrow):\ S=1} = 2\begin{pmatrix}1\\1\end{pmatrix} \qquad \begin{pmatrix}1&1\\1&1\end{pmatrix}\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{\frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow):\ S=0} = 0\begin{pmatrix}1\\-1\end{pmatrix}$$

* in general: $S_1 + S_2$ for arbitrary S_1, S_2



these 4 states (•) are linear combinations of these 4 states (x)

* the transformation matrix between these independent subspaces are called

* the transformation matrix between these independent subspaces are called "Clebsch-Gordon coefficients"

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TABLE 4.8: Clebsch-Gordan coefficients. (A square root sign is understood for every entry; the minus sign, if present, goes *outside* the radical.)

1/2 × 1/2

		1
		+1
+1/2	+1/2	1

		1	0
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		1
		−1
−1/2	−1/2	1

1 × 1/2

		3/2
		+3/2
+1	+1/2	1

		3/2	1/2
		+1/2	+1/2
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

		3/2	1/2
		−1/2	−1/2
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

		3/2
		−3/2
−1	−1/2	1

2 × 1

		3
		+3
+2	+1	1

		3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

		3	2	1
		+1	+1	+1
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	6/15	−1/2	1/10

		3	2	1
		0	0	0
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

		3	2	1
		−1	−1	−1
0	−1	6/15	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

		3	2
		−2	−2
−1	−1	2/3	1/3
−2	0	1/3	−2/3

		3
		−3
−2	−1	1

1 × 1

		2
		+2
+1	+1	1

		2	1
		+1	+1
+1	0	1/2	1/2
0	+1	1/2	−1/2

		2	1	0
		0	0	0
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

		2	1
		−1	−1
0	−1	1/2	1/2
−1	0	1/2	−1/2

		2
		−2
−1	−1	1

3/2 × 1

		5/2
		+5/2
+3/2	+1	1

		5/2	3/2
		+3/2	+3/2
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

		5/2	3/2	1/2
		+1/2	+1/2	+1/2
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

		5/2	3/2	1/2
		−1/2	−1/2	−1/2
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

		5/2	3/2
		−3/2	−3/2
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

		5/2
		−5/2
−3/2	−1	1

2 × 1/2

		5/2
		+5/2
+2	1/2	1

		5/2	3/2
		3/2	+3/2
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

		5/2	3/2
		+1/2	+1/2
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

		5/2	3/2
		−1/2	−1/2
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

		5/2	3/2
		−3/2	−3/2
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

		5/2
		−5/2
−2	−1/2	1

3/2 × 1/2

		2
		+2
+3/2	+1/2	1

		2	1
		+1	+1
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

		2	1
		0	0
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

		2	1
		−1	−1
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

		2
		−2
−3/2	−1/2	1