L75-Two Level Systems

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* Time-Dependent Perturbation Theory

- Quantum statics: $\hat{H} = \hat{T} + \hat{V}$ independent of time
  - $\Rightarrow$ TDSE separates into TISE (technique used till now)
  - can solve time dependence of wave function : $e^{-iE_n t/\hbar}$
  - but E_n still constant

- Quantum Dynamics: $\mathcal{H}(t)$
  - needed for transitions (quantum jumps)
  - example: atomic absorpion & emission of radiation

* 2-level system (unperturbed)

let $\mathcal{H} = \mathcal{H}^0 + \mathcal{H}'(t)$ $\quad \mathcal{H}^0 = \hat{T} + \hat{V}^0 \quad \mathcal{H}' =$ time-dependent perturbation potential

$\mathcal{H}^0 \psi_a = E_a \psi_a$ and $\mathcal{H}_0 \psi_b = E_b \psi_b \quad \Rightarrow \quad \langle \psi_i | \psi_j \rangle = \delta_{ij}$

$\psi(0) = c_a \psi_a + c_b \psi_b \quad \Rightarrow \quad \psi(t) = c_a \psi_a e^{-iE_a t/\hbar} + c_b \psi_b e^{-iE_b t/\hbar}$

normalization: $\langle \psi | \psi \rangle = |c_a|^2 + |c_b|^2 = 1$

* Perturbed system:

let $\psi(t) = c_a(t) \psi_a e^{-iE_a t/\hbar} + c_b(t) \psi_b e^{-iE_b t/\hbar}$ $\quad$ (ψ_a, ψ_b complete basis)

$$\mathcal{H}\psi(t) - i\hbar \frac{\partial}{\partial t}\psi = \left[\mathcal{H}^0 - i\hbar\frac{\partial}{\partial t}\right]\psi(t) + \mathcal{H}'(t)\,\psi(t) = 0$$

$$= \left(-i\hbar \dot{c}_a + c_a \mathcal{H}'\right)\psi_a e^{-iE_a t/\hbar} + \left(-i\hbar \dot{c}_b + c_b \mathcal{H}'\right)\psi_b e^{-iE_b t/\hbar} = 0$$

since $\left[\mathcal{H}^0 - i\hbar\frac{\partial}{\partial t}\right]\psi_a e^{-iE_a t/\hbar} = 0$, likewise for b.

matrix elements: take $\langle \psi_a |$ and $\langle \psi_b |$: $\quad \mathcal{H}'_{ij} = \langle \psi_i | \mathcal{H}' | \psi_j \rangle$

matrix elements: take $\langle\psi_a|$ and $\langle\psi_b|$: $H'_{ij} = \langle\psi_i|H'|\psi_j\rangle$

$$\langle\psi_a|-i\hbar\dot c_a + c_a H'|\psi_a\rangle e^{-iE_at/\hbar} + \langle\psi_a|-i\hbar \overset{0}{\cancel{\dot c_b}} + c_b H'|\psi_b\rangle e^{-iE_bt/\hbar} = 0$$

$$i\hbar\dot c_a = \overset{0}{\cancel{c_a H'_{aa}}} + c_b H'_{ab} e^{-i\omega_0 t} \qquad \text{where } \hbar\omega_0 = E_b - E_a$$

typically vanish

$$i\hbar \dot c_b = \overset{0}{\cancel{c_b H'_{bb}}} + c_a H'_{ab} e^{i\omega_0 t} \qquad i\hbar\begin{pmatrix}\dot c_a\\ \dot c_b e^{-i\omega_0 t}\end{pmatrix}=\begin{pmatrix}H'_{aa}&H'_{ab}\\H'_{ba}&H'_{bb}\end{pmatrix}\begin{pmatrix}\dot c_a\\ \dot c_b e^{-i\omega_0 t}\end{pmatrix}$$

this is an exact equation: TDSE in H_0 basis.

* Perturbation theory: $c_a^{(0)}, c_a^{(1)}, c_a^{(2)}, \dots$ successive approximations

stick $c_a^{(n)}, c_b^{(n)}$ into RHS, integrate $\int_0^t dt'\, i\hbar\dot c_{a,b}$ to get $c_{a,b}^{(n+1)}$ (LHS)

0th order: $c_a^{(0)}(t) = 1$ $\quad c_b^{(0)}(t) = 0$ [initial state a]

1st order: $i\hbar\dot c_a^{(1)} = \overset{0}{\cancel{c_b^{(0)}}} H'_{ab} e^{-i\omega t}$ $\qquad c_a = 1$

$$i\hbar \dot c_b^{(1)} = \overset{1}{\cancel{c_a^{(0)}}} H'_{ba} e^{i\omega_0 t} \qquad c_b^{(1)} = \frac{1}{i\hbar}\int_0^t dt'\, H'_{ba}(t) e^{i\omega_0 t'}$$

2nd order: $i\hbar \dot c_a^{(2)} = c_b^{(1)} H'_{ab} e^{i\omega_0 t}$ $\quad c_a^{(2)} = 1 - \frac{1}{\hbar^2}\int_0^t H'_{ab}(t) e^{-i\omega_0 t'}\left[\int_0^{t'} H'_{ba}(t'') e^{i\omega_0 t''} dt''\right]dt'$

* Sinusoidal Perturbations: $H' = V(\vec r)\cos(\omega t)$

$H'_{ab} = V_{ab}\cos\omega t$ where $V_{ab} = \langle\psi_a|V|\psi_b\rangle$ as usual

$$c_b^{(1)} = \frac{1}{i\hbar}\int_0^t dt\, V_{ba}\cos\omega t\, e^{i\omega_0 t} = \frac{V_{ba}}{2i\hbar}\int_0^t dt'\left(e^{i(\omega_0+\omega)t'} + e^{i(\omega_0-\omega)t'}\right)$$

$$= \frac{-V_{ba}}{2\hbar}\left[\overbrace{\cancel{\frac{e^{i(\omega_0+\omega)t}-e^0}{\omega_0+\omega}}}^{\text{small}} + \frac{e^{i(\omega_0-\omega)t}-e^0}{\omega_0-\omega}\right]$$

$$\simeq \frac{V_{ba}}{i\hbar}\,\frac{\sin\left(\frac{\omega_0-\omega}{2}t\right)}{\omega_0-\omega}\, e^{i\frac{\omega_0-\omega}{2}t}$$

$$P_{a\to b}(t) = |c_b^{(1)}|^2 = \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2\left(\frac{\omega-\omega_0}{2}t\right)}{(\omega-\omega_0)^2} \quad \text{for} \quad |V_{ab}|^2 \ll \hbar^2(\omega-\omega_0)^2$$

exact solution is called "Rabi flopping" (HW12)