L83-Born Approximation

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* general goal of attack: apply Green's functions to obtain a particular solution of Schrödinger equation with a "source" term. with a couple of hitches

* reminder: we already did this in E&M to solve Poisson's Eq.:

$$-\nabla^2 V = \rho/\varepsilon \quad \Rightarrow \quad V = -\nabla^{-2}\rho/\varepsilon \quad = \quad V_p + V_0$$

V_0 is the general solution of $\nabla^2 V = 0$ (homogenous eq.)

- the coefficients of V_0 are constrained to satisfy boundary conditions

V_p is a particular solution to $-\nabla^2 V = \rho/\varepsilon_0$ (one of many)

- we solve for V using Green's functions: the solution $-\nabla^2 G(\vec{\mathscr{r}}) = \delta^3(\vec{\mathscr{r}})$ with a basis function as the source.

$$4\pi G = \frac{1}{\mathscr{r}} \xrightarrow{-\nabla} \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \xrightarrow{\nabla\cdot} 4\pi\delta^3(\vec{\mathscr{r}}) \qquad V \xrightarrow{-\nabla} \vec{E} \xrightarrow{\nabla\cdot} \rho/\varepsilon \qquad \vec{\mathscr{r}} \equiv \vec{r} - \vec{r}'$$

- the solution is a linear combo of basis solutions indexed by $\vec{r}'$! (for a point charge at $\vec{r}'$): $\underbrace{\rho(\vec{r})}_{\text{vector}} = \int d^3r' \underbrace{\delta^3(\vec{r}-\vec{r}')}_{\text{basis}} \underbrace{\rho(\vec{r}')}_{\text{component}}$

$$V_p(\vec{r}) = -\nabla^{-2}\rho(\vec{r})/\varepsilon = -\nabla_{\vec{r}}^{-2}\left[\int d^3r'\, \delta^3(\vec{\mathscr{r}})\rho(\vec{r}')\right]/\varepsilon$$

$$= \int d^3r'\, \rho(\vec{r}')\left[-\nabla_{\vec{r}}^{-2}\delta^3(\vec{\mathscr{r}}) = G(\vec{\mathscr{r}}) = \frac{1}{4\pi\varepsilon\mathscr{r}}\right] = \int \frac{d^3r'\,\rho(\vec{r}')}{4\pi\varepsilon_0\mathscr{r}}$$

Example: Helmholtz theorem: $\nabla^2 = \nabla\nabla\cdot - \nabla\times\nabla\times$

$$\vec{F} = -\nabla\underbrace{\left(-\nabla^{-2}\overbrace{\nabla\cdot\vec{F}}^{\rho/\varepsilon}\right)}_{V} + \nabla\times\underbrace{\left(-\nabla^{-2}\overbrace{\nabla\times\vec{F}}^{\mu\vec{J}}\right)}_{\vec{A}}$$

$$-\nabla^2 V = \rho/\varepsilon \quad V = -\nabla^{-2}\rho/\varepsilon = \int\frac{d^3r'\,\rho(\vec{r}')}{4\pi\varepsilon\mathscr{r}} \qquad -\nabla^2\vec{A} = \mu\vec{J} \quad \vec{A} = \nabla^{-2}\mu\vec{J} = \int\frac{\mu_0}{4\pi}\frac{d^3r'\,\vec{J}(\vec{r}')}{\mathscr{r}}$$

* Application to Schrödinger Equation:

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V(\vec{r})\psi = E\psi \quad \xrightarrow{V=0} \quad \underbrace{(\nabla^2+k^2)\psi=0}_{\text{plane waves: homogenous solution}} \qquad E=\frac{\hbar^2k^2}{2m}$$

$$\underbrace{(\nabla^2+k^2)}_{\text{wave operator}}\psi = \underbrace{Q = \frac{2mV}{\hbar}\psi}_{\text{note that "source" term has } \psi(r) \text{ embedded!}}$$

Two "minor" differences: (latter causes major difficulties!)

a) need Green's function for Helmholtz, not Laplace eq'n:
$k^2 \approx -m^2$ called "mass term" : creates waviness or attenuation
note: $k^2=0$ for E&M: the photon is massless.

$$(\nabla^2+k^2)G(\vec{r}) = \delta^3(\vec{r}) \quad \text{where} \quad G(r) = -\frac{e^{ikr}}{4\pi r}$$

is solved in Fourier-momentum space: $\nabla e^{i\vec{q}\cdot\vec{r}} = i\vec{q}\, e^{i\vec{q}\cdot\vec{r}}$.

$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2}\int d^3r_0 \frac{e^{ik(\vec{r}-\vec{r}_0)}}{|\vec{r}-\vec{r}_0|} V(\vec{r}_0)\,\psi(\vec{r}_0)$$

b) the source term depends on the wave function:
use perturbation theory: $(\nabla^2+k^2)\psi^{(n)} = \frac{2mV}{\hbar^2}\psi^{(n-1)}(r)$
(the Born Series).

$$\psi_0 = e^{ikz} \rightarrow Q_0 \xrightarrow{(\nabla^2+k^2)^{-1}} \psi_1 \rightarrow Q_1 \xrightarrow{(\nabla^2+k^2)^{-1}} \psi_2 \; \dots$$