L84-Born Series

Wednesday, April 27, 2016 08:00

* Summary: last class we turned Schrödinger's PDE into an integral equation – solved perturbatively.

$$\underbrace{(\nabla^2+k^2)}_{\text{wave operator}}\psi(\vec r) = \frac{2mV}{\hbar^2}\psi(\vec r) \equiv \underbrace{Q(\vec r)}_{\text{source}} \quad \text{[TISE]}$$

$$\psi(\vec r) = (\nabla^2+k^2)^{-1}Q(\vec r) = \int d^3r_0\,[G_0+G](\vec r-\vec r_0)\cdot Q(\vec r_0)$$

$$= \psi_0(\vec r) + \int d^3\vec r_0\, \frac{e^{ik|\vec r-\vec r_0|}}{4\pi|\vec r-\vec r_0|}\frac{2m}{\hbar^2}V(\vec r_0)\,\psi(\vec r_0)$$

where $(\nabla^2+k^2)G_0(\vec r-\vec r_0)=0 \;\Rightarrow\; (\nabla^2+k^2)\psi_0(\vec r)=0$

ψ_0 is the solution to the homogenous equation, the arbitrary "constant of integration", used to satisfy the boundary conditions.
G_0 is a "free wave function", NOT a Green's function, because it has no source.

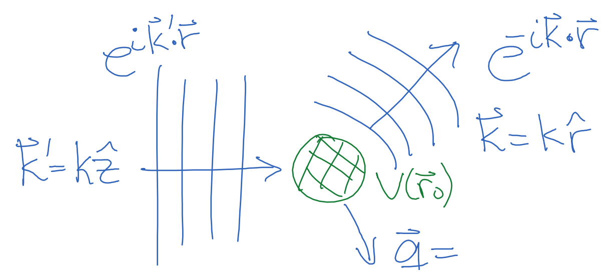
and $(\nabla^2+k^2)G(\vec r-\vec r_0) = \delta^3(\vec r-\vec r_0)$ $\qquad h_0^1(x) = \frac{-ie^{ix}}{x}$

$G = \frac{ik}{4\pi}h_0^{(1)}(kr) + N\,j_0(kr)$ are Green's functions for arbitrary N, since $(\nabla^2+k^2)j_0(kr)=0$ everywhere.

* Today we will:
  1) use the Born (PWIA) approximation to solve for the scattering amplitude & differential cross section
  2) give a physical interpretation of the expansion.

* Scattering amplitude: $\vec r\to\infty$

for a confined potential $V(\vec r_0)$,



for a confined potential $V(\vec{r}_0)$,

$|\vec{r}-\vec{r}_0|^2 \approx r^2 - 2\vec{r}\cdot\vec{r}_0 \approx r^2\left(1 - 2\frac{\vec{r}\cdot\vec{r}_0}{r^2}\right)$ so $|\vec{r}-\vec{r}_0| \approx r - \hat{r}\cdot\vec{r}_0$

let $k = k\hat{r}$ then $G = \frac{-e^{ik|\vec{r}-\vec{r}_0|}}{4\pi|\vec{r}-\vec{r}_0|} \approx \frac{-e^{ikr}}{4\pi r} e^{-i\vec{k}\cdot\vec{r}_0}$

Boundary conditions: $\psi_0 = Ae^{ikz}$ [incoming flux]

$$\psi(\vec{r}) \cong Ae^{ikz} + \underbrace{\left[\frac{-m}{2\pi\hbar^2}\int d^3\vec{r}_0\, e^{-i\vec{k}\cdot\vec{r}_0} V(\vec{r}_0)\, \psi(\vec{r}_0)\right]}_{f(\theta)} \frac{e^{ikr}}{r}$$

$f(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\langle e^{-i\vec{k}\cdot\vec{r}}|V(\vec{r})|\psi(\vec{r})\rangle$ where $\vec{k}$ points in (θ,ϕ) direction

This is the transition amplitude from $\psi(r)$ to $e^{i\vec{k}\cdot\vec{r}}$

* Born approximation: $\psi(\vec{r}_0) \approx \psi_0(\vec{r}_0) = Ae^{ikz} = Ae^{i\vec{k}'\cdot\vec{r}}$

where $\vec{k}' = k\hat{z}$ [Note: other texts reverse $k' \leftrightarrow \vec{k}'$!]

then $f(\theta,\phi) \approx \frac{-m}{2\pi\hbar^2}\langle e^{-i\vec{k}\cdot\vec{r}}|V(r_0)|e^{i\vec{k}'\cdot\vec{r}}\rangle = \frac{-m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}_0} V(\vec{r}_0)$

This is the Fourier transform of the potential!

$\hbar(\vec{q} \equiv \vec{k} - \vec{k}')$ is the momentum transfer

Low energy: $f(\theta,\phi) \approx \frac{-m}{2\pi\hbar^2}\int V(\vec{r})\, d^3r$

* Optical potential (Finite square well):

$V(r) = \begin{cases} V_0 & r<a \\ 0 & r>a \end{cases}$ $\quad f(\theta,\phi) = \frac{-m}{2\pi\hbar^2} V_0 \frac{4}{3}\pi a^3$

$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2 \quad \sigma = 4\pi\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2$

* Formally, $(\nabla^2+k^2)\psi = V\psi$ with $\frac{\hbar^2}{2m}$ absorbed into V.

$\psi = \psi_0 + GV\psi$, where $(\nabla^2+k^2)\psi_0 = 0$, $(\nabla^2+k^2)G = \delta^3$

$\psi = (1-GV)^{-1}\psi_0 = \psi_0 + GV\psi_0 + (GV)^2\psi_0 + (GV)^3\psi_0 + \ldots$

can also be obtained by itterating $\psi_n = \psi_0 + GV\psi_{n-1}$

$\psi_1 = \psi_0 + GV\psi_0$ $\psi_2 = \psi_0 + GV\psi_1 = \psi_0 + GV\psi_0 + GVGV\psi_0$, etc...

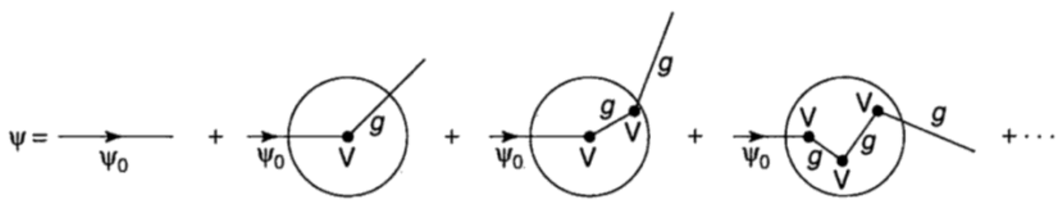


FIGURE 11.13: Diagrammatic interpretation of the Born series (Equation 11.101).