Exam 1 solution

Wednesday, March 9, 2016 07:22

[20 pts] 1. a) What are the physical implications of the commutation relations $\vec{L} \times \vec{L} = i\hbar\vec{L}$?

b) Show that $[L_z, L_\pm] = \pm\hbar L_\pm$, where $L_\pm \equiv L_x \pm iL_y$, using $[L_i, L_j] = \epsilon_{ijk} i\hbar L_k$.

c) Show that if $|\psi\rangle$ is an eigenvector of L_z with eigenvalue $\hbar m$, then $L_+|\psi\rangle$ is also an eigenvector of L_z, but with eigenvalue $\hbar(m+1)$.

d) What is the value of $L_+ Y_{\ell\ell}(\theta, \phi)$?

a) because $[L_x, L_y] = i\hbar L_z$, etc, it is impossible to know tw compoents of $\vec{L}$ simultaneously. by the H.U.P., $\Delta L_x \Delta L_y \geq \hbar L_z$. However, $[L_i, L^2] = 0$, and it is possible to specify simultaneous quantum numbers ℓ for L^2 and m for L_z. In other words, L^2, L_z are compatible, but not any two components of $\vec{L}$.

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b) $[L_z, L_\pm] = [L_z, L_x \pm iL_y] = [L_z, L_x] \pm i[L_z, L_y]$

6 $\quad = i\hbar L_y \pm i(-i\hbar L_x) = \pm\hbar(L_x \pm iL_y) = \pm\hbar L_\pm$

c) if $L_z|\psi\rangle = \hbar m|\psi\rangle$, then $L_z L_+ - L_+ L_z = \hbar L_+$

$L_z[L_+|\psi\rangle] = (L_z L_+)|\psi\rangle$ definition of operator composition

$= (L_+ L_z + \hbar L_+)|\psi\rangle$ from b)

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$= (L_+ \hbar m + \hbar L_+)|\psi\rangle$ use of eigenvalue

$= \hbar(m+1) L_+|\psi\rangle$

d) $L_+ Y_{\ell\ell}(\theta, \phi) = 0$ since $m = \ell$ is the highest level, but L_+ raises m by one.

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[20 pts] 2. a) Compare and contrast orbital ($\vec{L}$) and spin ($\vec{S}$) angular momentum.

b) How does the complex spinor $\chi = \begin{pmatrix} a \\ b \end{pmatrix}$ encode both directions (θ,ϕ) of spin? Which directions can be measured? What about the magnitude?

c) A spin-$\frac{1}{2}$ particle with magnetic moment $\vec{\mu} = \frac{1}{2}\hbar\gamma\vec{\sigma}$ precesses freely in a magnetic field $\vec{B} = \hat{x}B$ with total energy $\mathcal{H} = -\vec{\mu}\cdot\vec{B}$. Calculate the evolution of the state, $\chi(t)$, given the initial state $\chi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. (for -5 pts, you can use the field $\vec{B} = \hat{z}B$)

a) Both are angular momentum operators with the same symmetry: $L \times L = i\hbar L$ or $S \times S = i\hbar S$

Both generate rotations, in fact you need each to rotate a specific part ($\psi(\vec{r})$ or χ_{m_s}) of the wave function.

Both generate magnetic moments, but the Landé g-factor differs by a factor of 2.

Both have (j,m) quantum numbers, where $j \in (0, \frac{1}{2}, 1, \frac{3}{2}, \ldots)$ and $m = -j, -j+1, \ldots j-1, j$

L is a function of spatial coordinate, S acts on

5 spinor components, but both can be represented as matrices (any QM operator can!)

L corresponds to momentum of the wave packet, S to topologicaly invariant (intrinsic) momentum associated with the components of the spinor.

L can have any positive integer value: 0, 1, 2, 3, ... and has an associated wave: $Y_{lm}(\theta,\phi)$ in space.

S can have any half-integal value: 0, ½, 1, 3/2, ... but only one value (a property of the particle, like mass)

The both add together to form $J = L + S$, the total angular momentum, which is conserved.

b) The eigenvector of $\hat{n}\cdot\vec{\sigma}$ is $\begin{pmatrix} \cos\theta/2 \, e^{-i\phi/2} \\ \sin\theta/2 \, e^{i\phi/2} \end{pmatrix}$. In other words, the real part of a, b encode θ (spin up or spin down), and the phase difference between a and b encodes ϕ. The global

of spin down), and the phase difference
between a and b encodes ϕ. The global
3 phase is irrelevant, and $a^2+b^2=1$. Note the
half angles in the formula, a trademark of spin®.

While the spinor has an associated θ, ϕ, that is just the expectation value. $\hat{x}$ and $\hat{y}$ directions are linear combinations of $\hat{z}$ and $-\hat{z}$, so only the component along one direction can be known at any time. (uncertainty principle). The magnitude of $\vec{S}$ is always $\hbar/2$, and does not need to be encoded in χ.

c) $H = -\frac{1}{2}\hbar\gamma\vec{\sigma}\cdot\hat{x}B = -\frac{1}{2}\hbar\omega_L\sigma_x$ where $\omega_L = \gamma B$

TISE: $H\chi = -\frac{1}{2}\hbar\omega_L\sigma_x\chi = E\chi$

$$\sigma_x\chi = \lambda\chi \qquad \lambda = E/-\tfrac{1}{2}\hbar\omega_L$$

7 $$\begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \qquad \lambda = \pm 1$$

$$\lambda = +1: \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \lambda = -1: \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Thus the energies and eigenvectors are:

$$E_- = -\tfrac{1}{2}\hbar\omega_L, \quad \chi_- = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad E_+ = +\tfrac{1}{2}\hbar\omega_L \quad \chi_+ = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

TDSE: $\chi(t) = a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{-iE_-t/\hbar}$

initial conditions: $\chi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\chi_+ + b\chi_-$

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$$\chi_+^\dagger \chi_0 = \frac{1}{\sqrt{2}}(1\ -1)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}} = a\,\chi_+^\dagger \chi_+ + b\,\chi_+^\dagger \chi_- = a$$

$$\chi_-^\dagger \chi_0 = \frac{1}{\sqrt{2}}(1\ 1)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}} = a\,\chi_-^\dagger \chi_+ + b\,\chi_-^\dagger \chi_- = b$$

$$\chi(t) = \frac{1}{\sqrt{2}}\chi_+ e^{-iE_+t/\hbar} + \frac{1}{\sqrt{2}}\chi_- e^{-iE_-t/\hbar}$$

$$= \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}e^{+i\omega_L t/2} + \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix}e^{-i\omega_L t/2}$$

$$= \begin{pmatrix}\cos\omega_L t/2\\ -i\sin\omega_L t/2\end{pmatrix}$$

If $\vec{B} = \hat{z}B$, then the eigenvectors are

$$\sigma_z\begin{pmatrix}1\\0\end{pmatrix} = +\begin{pmatrix}1\\0\end{pmatrix} \qquad \sigma_z\begin{pmatrix}0\\1\end{pmatrix} = -\begin{pmatrix}0\\1\end{pmatrix}$$

and $\chi(t) = \begin{pmatrix}1\\0\end{pmatrix}e^{-i\omega_L t/2}$

[20 pts] 3. a) Given a parity operator P such that $P^2 = I$, and an arbitrary state ψ, show that the superposition $\psi_\pm \equiv \psi \pm P\psi$ has even/odd parity, ie. that $P\psi_\pm = \pm\psi_\pm$.

b) Specify and explain the configuration and spectroscopic term of the ground state of the lithium atom ($Z = 3$). What is the degeneracy of the ground state?

c) Write the exact wavefunction $\psi_0(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3)$ of the above ground state in terms of hydrogen-like ($Z = 3$) single-particle wave functions $\psi_{n\ell m_\ell m_s}$, ignoring electron pair interactions, but of course, taking into account the Pauli exclusion principle.

a) $P\psi_\pm = P(\psi \pm P\psi) = P\psi \pm P^2\psi = P\psi \pm \psi$

5 $\quad = \pm(\psi \pm P\psi) = \pm\psi_\pm$

b) There are 3 electrons, 2 in the closed 1s shell and the third in the 2s level. The total L, S in the closed shell is 0. The third electron has $l=0$ (s-orbital) and $s=1/2$, which
8 equal the totals L, S. $J = L+S = 0 + 1/2 = 1/2$ is the only option. The degeneracy is $2J+1 = 2$. (the third electron can be $\uparrow$ or $\downarrow$).

$_3Li: \quad (1s)^2(2s) \quad {}^2S_{1/2}$

c) $$\Psi(\vec{r}_1,\vec{r}_2,\vec{r}_3) = \frac{1}{\sqrt{6}} \begin{vmatrix} \psi_{1,0,0,1/2}(\vec{r}_1) & \psi_{1,0,0,1/2}(\vec{r}_2) & \psi_{1,0,0,1/2}(\vec{r}_3) \\ \psi_{1,0,0,-1/2}(\vec{r}_1) & \psi_{1,0,0,-1/2}(\vec{r}_2) & \psi_{1,0,0,-1/2}(\vec{r}_3) \\ \psi_{2,0,0,1/2}(\vec{r}_1) & \psi_{2,0,0,1/2}(\vec{r}_2) & \psi_{2,0,0,1/2}(\vec{r}_3) \end{vmatrix}$$

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which is completely antisymmetric, and has 3 particles,
one in $\psi_{1,0,0,1/2}$ $(n=1 \; l=0 \; m_l=0 \; m_s=\uparrow)$
one in $\psi_{1,0,0,-1/2}$ $(n=1 \; l=0 \; m_l=0 \; m_s=\downarrow)$
and one in $\psi_{2,0,0,+1/2}$ $(n=2 \; l=0 \; m_l=0 \; m_s=\uparrow)$
$\leftarrow$ alternatively we could have chosen $\downarrow$

[20 pts] 4. a) Given N particles in a system with only two energy levels $E_0 = 0$, and $E_1 = \epsilon$, each nondegenerate, calculate the total energy of the configuration (N_0, N_1).

b) Count the total number of microstates of the full N-body wave function for the configuration (N_0, N_1) above, assuming that the particles are i) distinguishable, ii) fermions, and iii) bosons. Explain your results.

c) Explain the distinctive features of ideal boson and fermion gasses at densities above the quantum concentration n_c. Include plots for illustration.

a) $E = \sum_n N_n E_n = N_0 \cdot 0 + N_1 \cdot \varepsilon = N_1 \varepsilon$

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b) distinguishable particles:
$Q = \binom{N}{N_0}$ the number of ways of choosing N_0 particles in the ground state, leaving N_1 particles in the excited state.

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The particles are distinuishable, but the states of the N_0 particles are identical thus the number of combinations, not permutations

fermions: only one particle can be in any state: so $Q(N_0,N_1)$ has the nonzero values $Q(0,0) = Q(1,0) = Q(0,1) = Q(1,1) = 1$. All others are $Q(N_0,N_1) = 0$. Thus $Q(N_0,N_1) = \binom{1}{N_0} \cdot \binom{1}{N_1}$

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bosons: any particle can be in any state: but the wave function must be symmetric, so there is only one state for each N_0, N_1: $Q(N_0, N_1) = 1$ for all N_0, N_1.

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c) Since fermions obey the Pauli exclusion principle, there can be at most one in any state. At zero temperature, they fill all the lowest energy states up to the Fermi energy E_F. At higher temperature, the highest energy particles can be excited up to slightly higher levels, approaching the classical distribution at higher temperature, where $n \ll n_c$.

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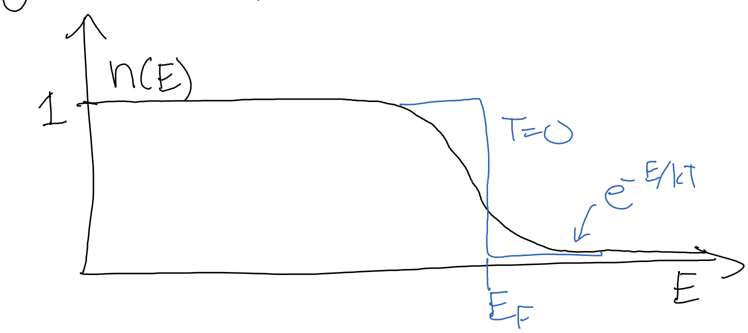

Bosons flock together, they are attracted to the same state, $n(E) \to \infty$ approaching $E \to \mu$. The distribution slides left/right to accomodate the number of particles N. Since $\mu < 0$, the maximum number of particles it can hold is when $\mu = 0$. The rest of the particles condense into the ground state as a BEC (Bose-Einstein Condensate).

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