University of Kentucky, Physics 521
Homework #13, Rev. A, due Thursday, 2017-02-02

0. Griffiths [2ed] Ch. 3 #39; Ch. 4 #27, #30, #31, #49, #52, #53.

1. Clifford algebra. The complete 3-vector algebra including dot and cross products can be implemented using the identity (I) as the unit scalar and Pauli matrices ($\boldsymbol{\sigma}$) as unit vectors:

$$1 = I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \hat{\boldsymbol{x}} = \boldsymbol{\sigma}_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\boldsymbol{y}} = \boldsymbol{\sigma}_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\boldsymbol{z}} = \boldsymbol{\sigma}_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

The dot and cross products can be represented by matrix multiplication, as in the formula

$$\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j = I(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) + i(\boldsymbol{\sigma}_i \times \boldsymbol{\sigma}_j) = I\delta_{ij} + i\epsilon_{ijk}\boldsymbol{\sigma}_k, \tag{2}$$

where the dot and cross products are interpreted in the usual sense of unit vectors. Note the difference between the imaginary i and the index $_i$. Also note that scalars are often implicitly multiplied by I when adding with other matrices. This algebra generalizes to a space-time algebra using the Dirac matrices γ^μ instead of the Pauli matrices σ_i.

a) Verify this formula for all nine products $\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j$ and show it is equivalent to the expression

$$(\boldsymbol{\sigma} \cdot \boldsymbol{a})(\boldsymbol{\sigma} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} + i\boldsymbol{\sigma} \cdot (\boldsymbol{a} \times \boldsymbol{b}). \tag{3}$$

Here, σ_x, σ_y, and σ_z are considered to be unit vectors as opposed to components of a vector $\boldsymbol{\sigma}$.

b) Which products are symmetric and which are antisymmetric? A general matrix product has both symmetric and antisymmetric parts, but this partition of symmetry into $i = j$ and $i \neq j$ is the defining feature of a Clifford algebra.

c) Show that any linear product $\boldsymbol{a} \circ \boldsymbol{b}$, can be decomposed into the sum $\boldsymbol{a} \circ \boldsymbol{b} = \{\boldsymbol{a} \circ \boldsymbol{b}\} + \langle \boldsymbol{a} \circ \boldsymbol{b} \rangle$ of symmetric $\{\boldsymbol{a} \circ \boldsymbol{b}\} \equiv \frac{1}{2}(\boldsymbol{a} \circ \boldsymbol{b} + \boldsymbol{b} \circ \boldsymbol{a})$ and antisymmetric $\langle \boldsymbol{a} \circ \boldsymbol{b} \rangle \equiv \frac{1}{2}(\boldsymbol{a} \circ \boldsymbol{b} - \boldsymbol{b} \circ \boldsymbol{a})$ parts, with respect to exchange of $\boldsymbol{a}$ and $\boldsymbol{b}$. Show that $\langle \boldsymbol{a} \circ \boldsymbol{a} \rangle = 0$ always. Why are the diagonal elements of an antisymmetric matrix zero? Apply this decomposition to the product $\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j$.

c) The imaginary i in the above formula is not present in the ordinary cross product. It distinguishes [axial] pseudovectors from [polar] vectors. Using part a), calculate the value of the pseudoscalar $\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j \boldsymbol{\sigma}_k$ and show it is completely antisymmetric in i, j, k. What is the analog of this triple product in terms of standard vector products?

2. Generators of rotation. In Griffiths #3.39, we showed that $p_x/\hbar$ is the generator of translation and $\mathcal{H}/\hbar$ is the generator of time evolution of the wavefunction: $\exp(-ip_x x_0/\hbar)\psi(x) = \psi(x - x_0)$ and $\exp(-i\mathcal{H}t_0/\hbar)\Psi(x,t) = \Psi(x, t + t_0)$. We also saw an example of this in Homework 9, where $M_z = \hat{\boldsymbol{z}}\times = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generates $R_\phi = \exp(M_z\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$, which rotates 2-dimensional vectors (spin 1, not two-component $s{=}\frac{1}{2}$ spinors). In 3-d, $\boldsymbol{M} = (M_x, M_y, M_z)$ generates the rotation $R_{\boldsymbol{\omega}} = \exp(\boldsymbol{M} \cdot \boldsymbol{\omega}) = I\cos\omega + \boldsymbol{M}{\cdot}\hat{\boldsymbol{\omega}}\sin\omega + \hat{\boldsymbol{\omega}}\hat{\boldsymbol{\omega}}^T(1 - \cos\omega)$ of 3-vectors about the axis $\boldsymbol{\omega}$.

a) In analogy with p_x, show that L_z generates the rotation of a wave function about the z-axis: $\exp(-i\phi_0 L_z/\hbar)\psi(r,\theta,\phi) = \psi(r,\theta,\phi - \phi_0)$. This generalizes to $\exp(i\boldsymbol{L} \cdot \boldsymbol{\omega})\psi(\boldsymbol{r}) = \psi(R_{\boldsymbol{\omega}}\boldsymbol{r})$, where $R_{\boldsymbol{\omega}}$ is a normal rotation matrix for vectors.

b) Show that the generators $\boldsymbol{M}$ are the cartesian equivalent of the 3×3 spin s=1 generator matrices $\boldsymbol{i}\boldsymbol{S}/\hbar$ in spherical tensor components of Griffiths #4.31. Hint: The vector $\boldsymbol{v} = (v_x, v_y, v_z)$ has spherical tensor components $v_{\pm 1} = \frac{1}{\sqrt{2}}(v_x \pm i v_y)$ and $v_0 = v_z$, which are not the same as its spherical components $\boldsymbol{v} = \hat{\boldsymbol{r}} v_r + \hat{\boldsymbol{\theta}} v_\theta + \hat{\boldsymbol{\phi}} v_\phi$.

c) Calculate the $s = \frac{1}{2}$ spinnor rotation matrices $R_i(\phi) = \exp(i\sigma_i\phi/2)$ about the $i = x, y, z$ axes. How do these relate to the eigenvectors of $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}$ in Griffiths #3.30? Show that a spinnor changes sign after a full revolution and must be rotated by 4π to return back to its original value.