University of Kentucky, Physics 521
Homework #14, Rev. A, due Friday, 2017-02-10

0. Griffiths [2ed] Ch. 4 #28, #58, #59, #61.

1. Neutron in a magnetic field–the neutron is spin-$\frac{1}{2}$ with magnetic moment $\mu_n = -1.91\ \mu_N$ in units of the *nuclear magneton* $\mu_N = \frac{e\hbar}{2m_p}$. The corresponding operator is $\boldsymbol{\mu} = \mu_n \boldsymbol{\sigma}$. The gyromagnetic factor is $\gamma_n = g_s \mu_N/\hbar$, where the Landé factor is $g_s = 2\mu_n$. Due to spin precession in the magnetic field, both the classical and quantum mechanical equations of motion are greatly simplified in a rotating reference frame, where $(\hat{\boldsymbol{x}}' \quad \hat{\boldsymbol{y}}') = (\hat{\boldsymbol{x}} \quad \hat{\boldsymbol{y}}) \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}$. [Rabi, Ramsey, Schwinger, Rev. Mod. Phy. **26**, 167, 1954]

a) Solve the classical equation of motion [see our class notes]

$$\frac{d\boldsymbol{S}}{dt} = \boldsymbol{\tau} = \boldsymbol{\mu}_n \times \boldsymbol{B} = \gamma_n \boldsymbol{S} \times \boldsymbol{B} \tag{1}$$

for *Larmor precession* of a neutron in a constant magnetic field $\hat{\boldsymbol{z}} B_0$, with initial spin $\boldsymbol{S}_0$ at $t = 0$. Using the following steps, solve for the same motion in the rotating frame: since the operator $\boldsymbol{\omega} dt \times$ generates rotation, $\frac{d\boldsymbol{S}}{dt} = \frac{d\boldsymbol{S}'}{dt} + \boldsymbol{\omega} \times \boldsymbol{S}$, where $\frac{d\boldsymbol{S}'}{dt}$ is with respect to components in the rotating frame. Substite $\frac{d\boldsymbol{S}}{dt}$ into Eq. 1 and show that the form remains the same except for the replacement of $\boldsymbol{B}$ with the effective field $\boldsymbol{B}' = \boldsymbol{B} + \boldsymbol{\omega}/\gamma_n$. Note this field is zero if the frame is rotating at the Larmor frequency $\boldsymbol{\omega}_L = -\gamma_n \boldsymbol{B}$, and thus the spin remains constant $\boldsymbol{S}' = \boldsymbol{S}_0$. Reconcile this with the solution in the static frame.

b) Solve the same precession quantum mechanically [see our class notes] starting from the initial state $\chi_0 = \begin{pmatrix} a \\ b \end{pmatrix}$. Show that the expected value of spin $\langle \boldsymbol{S} \rangle(t)$ agrees with the classical result. Using the results of Griffiths Ch. 4 #56, show that $\chi(t) = \exp(-i\boldsymbol{\omega} \cdot \boldsymbol{S}\, t\hbar)\chi'(t)$ for χ' in the rotating frame. Show that the same effective field $\boldsymbol{B}' = \boldsymbol{B} + \gamma_n \boldsymbol{\omega}$ as above appears in the Hamiltonian in the rotating frame if $\boldsymbol{B}$ is parallel to $\boldsymbol{\omega}$. Thus the state remains constant in the frame that rotates at the Larmor frequency. Show that this is equivalent to the solution in the static frame.

c) Note that the spin state m does not change in a constant magnetic field $\hat{\boldsymbol{z}} B_0$. Because it maintains the spin state, it is called a *holding field.* To make a transition, we use an oscillatory (RF) field $B_1(\hat{\boldsymbol{x}} \cos\omega t + \hat{\boldsymbol{y}} \sin\omega t)$. In the rotating frame with angular velocity $\hat{\boldsymbol{z}}\omega$, show that the total field is $\boldsymbol{B}' = \hat{\boldsymbol{z}}'(B_0 + \omega/\gamma_n) + \hat{\boldsymbol{x}}' B_1$, which is constant. Let θ be the angle between $\boldsymbol{B}'$ and $\hat{\boldsymbol{z}}$, and let the initial spin be $\boldsymbol{S}_0 = \hat{\boldsymbol{z}}$. Show classically that the z-component of the spin varies as $S_z(t) = S_0(\cos^2\theta + \sin^2\theta \cos\gamma_n B_1 t) = S_0(1 - 2\sin^2\theta \sin^2(\gamma_n B_1 t/2))$, which oscillates at the *Rabi flopping frequency* $\gamma_n B_1$. Plot the amplitude of oscillations as a function of the RF frequency ω and note the resonance at $\omega = \omega_L$. Thus this is called a resonant transition, and is the basis of *nuclear magnetic resonance* (NMR), a technique used in MRI machines. Plot the 3-d trajectory of $\boldsymbol{S}$ in the lab frame over half a Rabi cycle.

d) Solve the same problem quantum mechanically to show that the probability $|a(t)|^2$ of measuring spin up at time t, is the same as the classical result $S_z(t)/S_0$.