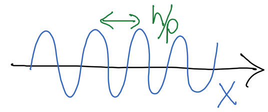
L51-Angular Momentum

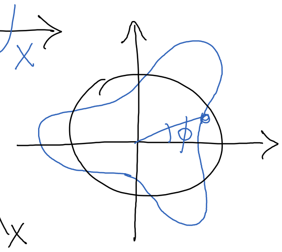
Friday, January 13, 2017 08:29

* What is the relation between p & x ?

- de Broglie wavelength: $p_x = \hbar k$

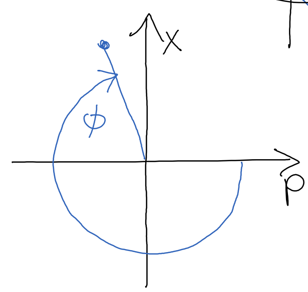


- Bohr correspondence: $L_z = \hbar m$ vs. ϕ



- Pauli uncertainty: $[p_x, x] = i\hbar$
- Phase-space (action-angle)

  $\Delta p \Delta x$ has units of $E \cdot t \equiv$ action



- Lagrangian - Hamiltonian mechanics

  $p\dot{x} = mv^2$ [energy] $\quad \dot{p}x = F \cdot x = \text{work} \quad E\dot{t} = \text{energy}.$

  $$\mathcal{L} = T - V = \tfrac{1}{2}mv^2 - V \qquad p \equiv \frac{\partial \mathcal{L}}{\partial \dot{x}} \text{ [cannonical momentum]}$$

  $$\mathcal{H} = p\dot{x} - \mathcal{L} = T + V = \frac{p^2}{2m} + V \qquad \dot{x} = \frac{\partial \mathcal{H}}{\partial p} \text{ [cannonical conjugates]}$$

- Angular momentum: cannonical conjugate of angle!

  $$\mathcal{L} = \tfrac{1}{2}mv^2 = \tfrac{1}{2}mr^2\dot{\phi}^2 \qquad p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi} = mvr = L_z$$

* We have seen angular momentum in separation of variables:

$$\hat{T} = \frac{\hat{p}^2}{2m} = \frac{-\hbar^2}{2m}\nabla^2 = \frac{-\hbar^2}{2m}\frac{1}{r}\frac{\partial^2}{\partial r^2}r + \frac{1}{2mr^2}\Big[\underbrace{\frac{-\hbar^2}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\Big(\underbrace{-i\hbar\frac{\partial}{\partial\phi}}_{L_z = \hbar m}\Big)^2}_{L^2 = \hbar^2 l(l+1)}\Big] = \frac{\hat{p}_r^2}{2m} + \frac{\hat{L}^2}{2I}$$

Today we will show that this is really angular momentum and resolve for the eigenvalues using operator methods.

This turns out to be the only way to analyze generalized angular momentum like spin.

* Definition of quantum mechanical angular momentum:

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$$L = \vec{r} \times \vec{p} = [y p_z - z p_y,\ z p_x - x p_z,\ x p_y - y p_x] = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$L_z = -i\hbar \frac{\partial}{\partial \phi} = -i\hbar \left(\frac{\partial x}{\partial \phi} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \phi} \frac{\partial}{\partial y} \right) = -y p_x + x p_y$$

using $x = \rho \cos\phi$ $\quad \frac{\partial x}{\partial \phi} = -y \qquad y = \sin\phi \quad \frac{\partial y}{\partial \phi} = x$

* commutation relations: cannonical relations $\boxed{[r_i, p_j] = i\hbar\, \delta_{ij}}$

$$[L_x, L_y] = [y p_z - z p_y,\ z p_x - x p_z]$$

$$= [y p_z, z p_x] + [y p_z, -x p_z] + [-z p_y, z p_x] + [-z p_y, -x p_z]$$

$$= y [p_z, z] p_x + x [z, p_z] p_y = i\hbar (-y p_x + x p_y) = i\hbar L_z$$

(using the identities:

$$[A, BC] = A(BC) - (BC)A = B(AC - CA) + (AB - BA)C = B[A,C] + [A,B]C$$

$$[XY, Z] = -[Z, XY] = -X[Z,Y] - [Z,X]Y = [X,Z]Y - X[Y,Z]$$)

by cyclic permutation, $\boxed{[L_i, L_j] = \varepsilon_{ijk}\, i\hbar L_k}$

or formally, $\boxed{\hat{L} \times \hat{L} = i\hbar \hat{L}}$

this forms a generalized definition of angular momentum!

* define $\hat{L}^2 \equiv L_x^2 + L_y^2 + L_z^2$

then $$[L_x, L^2] = [L_x, L_x^2] + [L_x, L_y^2] + [L_x, L_z]^2$$

$$= [L_x, L_y] L_y + L_y [L_x, L_y] + [L_x, L_z] L_z + L_z [L_x, L_z]$$

$$= i\hbar (L_z L_y + L_y L_z) - i\hbar (L_y L_z + L_z L_y) = 0$$

thus $\boxed{[L^2, \hat{L}] = 0}$

we can choose L^2, L_z to be a complete set of commuting operators over the space of angular functions. (these appear in ∇^2!)

$*$ factor $L^2 = L_x^2 + L_y^2 + L_z^2 \overset{?}{=} (L_x + iL_y)(L_x - iL_y) + L_z^2$ NO! commutation relations!

define: $L_\pm \equiv L_x \pm i L_y$ $\quad L_\pm^\dagger = L_\mp$ ladder operators $\quad [L_\pm, L^2] = 0$ still.

$$L_+L_- = (L_x + iL_y)(L_x - iL_y) = L_x^2 + L_y^2 - i[L_x, L_y] = L_x^2 + L_y^2 + \hbar L_z$$

$$L_-L_+ = L_x^2 + L_y^2 - \hbar L_z \quad \text{so} \quad [L_+, L_-] = 2[L_x, L_y] = 2\hbar L_z$$

$$[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y] = i\hbar(L_y \mp i L_x) = \pm\hbar L_\pm$$

thus $\quad L_z L_\pm = (L_\pm \pm \hbar) L_z \quad$ it raises or lowers eigenvalues

$*$ summary:

		CSCO		B		ladders		
	$[A,B]$	L^2	L_z	L_x	L_y	L_+	L_-	
	L^2	0	0	0	0	0	0	
	L_z	0	0	$i\hbar L_y$	$-i\hbar L_x$	$\pm\hbar$		ladders.
A	L_x	0	$-i\hbar L_y$	0	$i\hbar L_z$	$\pm\hbar L_z$		
	L_y	0	$i\hbar L_x$	$-i\hbar L_z$	0	$-i\hbar L_z$		
	L_+	0		(antisgn)		0	$2\hbar L_z$	factorization.
	L_-	0					0	