L55-Magnetic Moment

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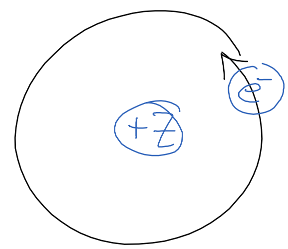
* gyromagnetic factor $\vec{\mu} = \gamma \vec{J}$

- consider charge/mass orbiting a nuclei



$$\gamma = \frac{\mu = Ia = \lambda v \cdot a = \frac{-e}{2\pi r} v \cdot \pi r^2}{L = r \times p = mrv} = \boxed{\frac{-e}{2m}}$$

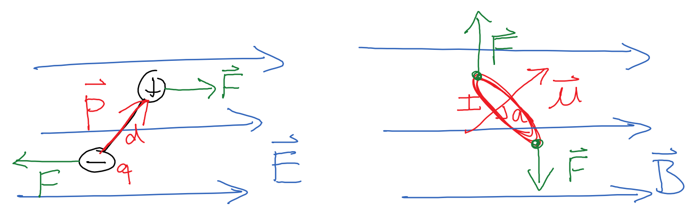
- in general $\gamma = g_L \frac{e}{2m}$ if charge and mass distribution differ

- Bohr magneton: $\mu_B \equiv \frac{e\hbar}{2m_e}$ so $\mu_z = g_L \mu_B \cdot m$

  nuclear magneton: $\mu_N = \frac{e\hbar}{2m_p}$

- the same formula holds for spin, except $\boxed{g_s = 2}$! $\mu_z = \pm\mu_B$
  (not charge/mass orbiting, but still close)

* dipole dynamics

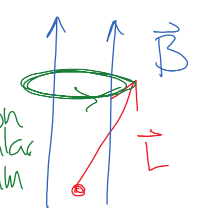


$$\vec{F} = q\vec{E} \rightarrow \vec{\tau} = \vec{p} \times \vec{E}$$

$$\boxed{\vec{\tau} = \vec{\mu} \times \vec{B}}$$

$$W = \int \tau \, d\theta = \boxed{-\vec{\mu} \cdot \vec{B}} \qquad \vec{F} = -\nabla W = \boxed{\nabla(\vec{\mu} \cdot \vec{B})}$$

* example: spin precession



– classical: $\dot{\vec{L}} = \vec{\tau} = \vec{\mu} \times \vec{B} = \gamma \vec{L} \times \vec{B}$

let $\vec{B} = \hat{z} B$ then $\dot{L}_z = 0$; $\dot{L}_x = \gamma B L_y$ $\dot{L}_y = -\gamma B L_x$

let $L_+ \equiv L_x + i L_y$ then $\dot{L}_+ = -i\gamma B L_+$ $L_+ = L_+^\circ e^{-i\gamma B t}$

- Angular momentum processes at the Larmor frequency $\omega_L = \gamma B$

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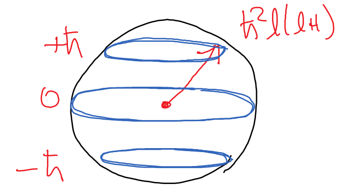
- quantum mechanically: the same thing happens



note: there are only fixed m-levels, but the changing phase is associated with precession.

$$\mathcal{H} = -\gamma_0 B S_z = -\frac{\gamma_0 B\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

eigenstates $\chi_+ = \begin{pmatrix}1\\0\end{pmatrix}$, $\chi_- = \begin{pmatrix}0\\1\end{pmatrix}$ with energy $E_\pm = \mp\frac{\gamma B\hbar}{2}$

$$\chi(t) = a\,\chi_+ e^{-iE_+t/\hbar} + b\,\chi_- e^{-E_-t/\hbar} = \begin{pmatrix}a\,e^{i\gamma Bt/2}\\ b\,e^{-i\gamma Bt/2}\end{pmatrix} \qquad \chi(0) = \begin{pmatrix}a\\b\end{pmatrix}$$

let $a = \cos(\alpha/2)$ $b = \sin(\alpha/2)$
so $|a|^2 + |b|^2 = 1$

$$\chi(t) = \begin{pmatrix}\cos(\alpha/2)\,e^{i\gamma Bt/2}\\ \sin(\alpha/2)\,e^{-i\gamma Bt/2}\end{pmatrix}$$

$$\langle S_x\rangle = \langle\chi|S_x|\chi\rangle = \frac{\hbar}{2}\left(\cos\tfrac{\alpha}{2}e^{-i\gamma Bt/2}\ \sin\tfrac{\alpha}{2}e^{i\gamma Bt/2}\right)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}\cos(\alpha/2)\,e^{i\gamma Bt/2}\\ \sin(\alpha/2)\,e^{-i\gamma Bt/2}\end{pmatrix}$$

$$= \frac{\hbar}{2}\,2\cdot\cos\alpha/2\cdot\sin\alpha/2\cdot\frac{1}{2}\left(e^{i\gamma Bt} + e^{-i\gamma Bt}\right) = \frac{\hbar}{2}\boxed{\sin\alpha\,\cos(\gamma Bt)}$$

$$\langle S_y\rangle = \langle\chi|S_y|\chi\rangle = \frac{\hbar}{2}\left(\cos\tfrac{\alpha}{2}e^{-i\gamma Bt/2}\ \sin\tfrac{\alpha}{2}e^{i\gamma Bt/2}\right)\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}\begin{pmatrix}\cos(\alpha/2)\,e^{i\gamma Bt/2}\\ \sin(\alpha/2)\,e^{-i\gamma Bt/2}\end{pmatrix}$$

$$= \frac{\hbar}{2}\,2\cdot\cos\alpha/2\cdot\sin\alpha/2\cdot\frac{-1}{2i}\left(e^{i\gamma Bt} - e^{-i\gamma Bt}\right) = \boxed{-\frac{\hbar}{2}\sin\alpha\,\sin(\gamma Bt)}$$

$$\langle S_z\rangle = \langle\chi|S_z|\chi\rangle = \frac{\hbar}{2}\left(\cos\tfrac{\alpha}{2}e^{-i\gamma Bt/2}\ \sin\tfrac{\alpha}{2}e^{i\gamma Bt/2}\right)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos(\alpha/2)\,e^{i\gamma Bt/2}\\ \sin(\alpha/2)\,e^{-i\gamma Bt/2}\end{pmatrix}$$

$$= \frac{\hbar}{2}\left(\cos^2\alpha/2 - \sin^2\alpha/2\right) = \boxed{\frac{\hbar}{2}\cos\alpha}$$

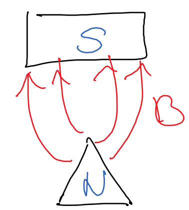
constant of the motion: $[H, S_z] = 0$

* example: Stern-Gerlach spin filter

- spin-dependent steering in a quadrupole field. better yet, a sextupole has a constant gradient everywhere inside

- this is one of the predominant methods of polarizing protons, neutrons, and nuclei. : ABS atomic beam source.

quadrupole scalar potential: $U = B_0 z + \frac{1}{2}\alpha(z^2 - x^2)$



$$\vec{B} = -\hat{x}\alpha x + \hat{z}(B_0 + \alpha z)$$

$$\vec{F} = \nabla(\vec{\mu}\cdot\vec{B}) = \nabla(\mu_z(B_0 + \alpha z)) = \hat{z}\alpha\mu_z = \hat{z}\alpha\gamma S_z$$

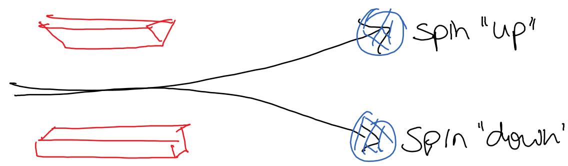
if we turn the force $\vec{F}$ on for time T,

$$\mathcal{H} = -\gamma(B_0 + \alpha z)S_z \quad \theta(0 \le t < T)$$

then $\chi(t) = a\chi_+ e^{-iE_+t/\hbar} + b\chi_- e^{+iE_-t/\hbar}$, where $E_\pm = \mp\gamma(B_0 + \alpha z)\frac{\hbar}{2}$

$$= (ae^{i\phi}\chi_+)e^{i\phi z} + (be^{i\phi}\chi_-)e^{-i\phi z} \quad \text{where } \phi = \gamma B T/2$$

but $\phi_{p_z}(z) = e^{ip_z z/\hbar}$, thus $p_\pm = \hbar\phi = \hbar\gamma B T/2$



* what "direction" are the two components of a spinor associated?

- a spinor actually has 4 real components (2 complex) except normalization & global phase factor
- thus it can represent 2 directions of $\hat{n} = (s_\theta c_\phi, s_\theta s_\phi, c_\theta)$ but not magnitude.
- rephrase question: what spinors $\chi_{n\pm}$ have spin "direction" $\hat{n}$?

$$\langle\vec{S}\rangle = \hbar/2\langle\vec{\sigma}\rangle = \hbar/2\langle\chi_{n+}|\vec{\sigma}|\chi_{n-}\rangle = \pm\hbar/2\,\hat{n}$$

- related question: what are the eigenvectors of $\sigma_n \equiv \hat{n}\cdot\vec{\sigma}$?

$$\langle\chi_{n\pm}|\sigma_n|\chi_{n\pm}\rangle = \pm\hat{n}\cdot\hat{n} = \pm 1 \quad \text{or} \quad \sigma_n\chi_{n\pm} = \pm\chi_{n\pm}$$

$$\sigma_n = \vec{\sigma}\cdot\hat{n} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} S_\theta C_\phi + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} S_\theta S_\phi + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} C_\theta = \begin{pmatrix} C_\theta & S_\theta e^{-i\phi} \\ S_\theta e^{i\phi} & -C_\theta \end{pmatrix}$$

$$|\sigma_n - \lambda I| = \begin{vmatrix} C_\theta - \lambda & S_\theta e^{-i\phi} \\ S_\theta e^{i\phi} & -C_\theta - \lambda \end{vmatrix} = \lambda^2 - (C_\theta^2 + S_\theta^2) = \lambda^2 - 1 = 0$$

thus $\lambda = \pm 1$

let $\theta = 2\alpha$, $\phi = 2\beta$

$C_\theta = C_{2\alpha} = C_\alpha^2 - S_\alpha^2 = 1 - 2S_\alpha^2 = 2C_\alpha^2 - 1$

$S_\theta = S_{2\alpha} = 2S_\alpha C_\alpha \qquad 1 + C_\theta = 2C_\alpha^2 \qquad 1 - C_\theta = 2S_\alpha^2$

$$\lambda = +1: \begin{pmatrix} C_\theta - 1 & S_\theta e^{-i\phi} \\ S_\theta e^{i\phi} & -C_\theta - 1 \end{pmatrix} \begin{pmatrix} \chi_+^{(n)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \chi_{n+} = \begin{pmatrix} S_\theta e^{-i\phi} \\ 1 - C_\theta \end{pmatrix} = \begin{pmatrix} 2S_\alpha C_\alpha e^{-i\phi} \\ 2S_\alpha^2 \end{pmatrix} \propto \begin{pmatrix} C_{\theta/2} e^{-i\phi/2} \\ S_{\theta/2} e^{i\phi/2} \end{pmatrix}$$

$$\lambda = -1: \begin{pmatrix} C_\theta + 1 & S_\theta e^{-i\phi} \\ S_\theta e^{i\phi} & -C_\theta + 1 \end{pmatrix} \begin{pmatrix} \chi_-^{(n)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \chi_-^{(n)} = \begin{pmatrix} -S_\theta e^{-i\phi} \\ 1 + C_\theta \end{pmatrix} = \begin{pmatrix} -2S_\alpha C_\alpha e^{-i\phi} \\ 2C_\alpha^2 \end{pmatrix} \propto \begin{pmatrix} -S_{\theta/2} e^{-i\phi/2} \\ C_{\theta/2} e^{i\phi/2} \end{pmatrix}$$

$$U_{(n)} = \left(\chi_{n+} \,\vdots\, \chi_{n-}\right) = \begin{pmatrix} C_{\theta/2} e^{-i\phi/2} & -S_{\theta/2} e^{-i\phi/2} \\ S_{\theta/2} e^{i\phi/2} & C_{\theta/2} e^{i\phi/2} \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} C_{\theta/2} & -S_{\theta/2} \\ S_{\theta/2} & C_{\theta/2} \end{pmatrix}}_{R} = ER$$

notation: $\chi_A^{(b)} = U_{(a)}^{(b)} \chi_A^{(a)}$ $\qquad$ (a), (b) identify associated basis. (default: (z))

A labels a particular spinor (ex. eig. χ_+)

check unitarity: $U^\dagger U = R^T E^* E R = R^T R = 1$

later we check that $U^\dagger \vec{\sigma} U = \begin{pmatrix} \hat{n} & ? \\ ? & -\hat{n} \end{pmatrix}$

note the half-angles! "spinor = $\sqrt{\text{rotation}}$"

$$U_{(x)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \underbrace{\begin{matrix} 1 \\ 1 \end{matrix}}_{\chi_{x+}} & \underbrace{\begin{matrix} -1 \\ 1 \end{matrix}}_{\chi_{x-}} \end{pmatrix} \qquad U_{(y)} = \frac{1}{\sqrt{2}} \begin{pmatrix} \underbrace{\begin{matrix} 1 \\ i \end{matrix}}_{\chi_{y+}} & \underbrace{\begin{matrix} -1 \\ i \end{matrix}}_{\chi_{y-}} \end{pmatrix} \qquad U_{(z)} = \begin{pmatrix} \underbrace{\begin{matrix} 1 \\ 0 \end{matrix}}_{\chi_{z+}} & \underbrace{\begin{matrix} 0 \\ 1 \end{matrix}}_{\chi_{z-}} \end{pmatrix}$$

ie $U_{(z)}^{(z)} = I$