L57-Two-Particle Systems

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* with 2-particle states, independent wave functions $\psi_1(\vec{r}_1)$, $\psi_2(\vec{r}_2)$ don't carry enough information (about correlations).

we need a joint wave function: $\Psi(\vec{r}_1, \vec{r}_2, t)$

$$\mathcal{H} = \frac{-\hbar^2}{2m}\nabla_1^2 + \frac{-\hbar^2}{2m}\nabla_2^2 + V(\vec{r}_1, \vec{r}_2, t) \qquad \mathcal{H}\Psi = \hat{E}\Psi = i\hbar\partial_t\Psi$$

- $|\Psi^2(\vec{r}_1, \vec{r}_2, t)|$ = "probability that particle 1 is at $\vec{r}_1$ and " " 2 $\vec{r}_2$ at time t."

$$\int d^3\vec{r}_1\, d^3\vec{r}_2\, |\Psi(\vec{r}_1, \vec{r}_2, t)|^2 = 1 \qquad \text{– total probability} = 1$$

$$\int d^3\vec{r}_2 |\Psi(\vec{r}_1, \vec{r}_2, t)|^2 = |\psi(\vec{r}_1, t)|^2 \qquad \text{– marginal probability of part. 1}$$

- Independent probabilities: $\Psi(\vec{r}_1, \vec{r}_2, t) = \Psi_1(\vec{r}_1, t) \cdot \Psi_2(\vec{r}_2, t)$

- separate time dependence as usual: $\hat{E}\Psi = E\Psi$ $\quad \Psi = \psi e^{-iEt/\hbar}$

$$\Psi(\vec{r}_1, \vec{r}_2, t) = \psi(\vec{r}_1, \vec{r}_2)\, e^{-iEt/\hbar} \qquad \hat{\mathcal{H}}\,\psi(\vec{r}_1, \vec{r}_2) = E\,\psi(\vec{r}_1, \vec{r}_2)$$

* Prob #5.1 Reduced mass:

- often the force only depends on $\vec{r} \equiv \vec{r}_1 - \vec{r}_2$ (central potential)

- then the center of mass (& momentum) moves as a free particle. $\quad M\vec{R} \equiv m_1\vec{r}_1 + m_2\vec{r}_2$, $\quad M \equiv m_1 + m_2$

- we can describe the system as two independent "particles":

centre of mass: $(M, \vec{R})$ & reduced mass (relative) $(\mu, \vec{r})$

$$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 \qquad m_1\vec{r}_1 = m_1\vec{R} + \mu\vec{r}$$

$$(\ \vec{r} = \vec{r}_1 - \vec{r}_2\ ) \times m_2, -m_1 \qquad m_1\vec{r}_2 = m_1\vec{R} - \mu\vec{r}$$

$$M\vec{R} = m_1\vec{r}_1 + m_2\vec{r}_2 \qquad\qquad m_1\vec{r}_1 = m_1\vec{R} + \mu\,\vec{r}$$

$$(\ \vec{r} = \vec{r}_1 - \vec{r}_2\ )\times m_2,\ -m_1 \qquad\qquad m_1\vec{r}_2 = m_1\vec{R} - \mu\,\vec{r}$$

where $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$ $\quad \mu \equiv \frac{m_1 m^2}{m_1+m_2}$ is the reduced mass.

$$\vec{P} = m_1\dot{\vec{r}}_1 + m_2\dot{\vec{r}}_2 = m_1\dot{\vec{R}} + \underbrace{\cancel{\frac{m_1 m_2}{M}}}_{\mu}\dot{\vec{r}} + m_2\dot{\vec{R}} - \underbrace{\cancel{\frac{m_1 m_2}{M}}}_{\mu}\dot{\vec{r}} = M\dot{\vec{R}}$$

$$T = \tfrac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \tfrac{1}{2}m_2\dot{\vec{r}}_2^{\,2} = \tfrac{1}{2}m_1\left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right)^2 + \tfrac{1}{2}m_2\left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right)^2$$

$$= \tfrac{1}{2}(m_1+m_2)\dot{\vec{R}}^2 + \tfrac{1}{2}\frac{m_1 m_2(m_1+m_2)}{M^2}\dot{\vec{r}}^{\,2} = \underbrace{\tfrac{1}{2}M\dot{\vec{R}}^2}_{\text{CM particle}} + \tfrac{1}{2}\mu\dot{\vec{r}}^{\,2}$$

"reduced" relative particle.

- Hamiltonian likewise splits: $\quad m_1 x_1 + m_2 x_2 = MX$, $\quad x_1 - x_2 = x$

$$\frac{\partial}{\partial x_1} = \frac{\partial X}{\partial x_1}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_1}\frac{\partial}{\partial x} = \frac{m_1}{M}\frac{\partial}{\partial X} + \frac{\partial}{\partial x} \;\rightarrow\; \nabla_1 = \frac{m_1}{M}\nabla_{\vec{R}} + \nabla_{\vec{r}}$$

$$\frac{\partial}{\partial x_2} = \frac{\partial X}{\partial x_2}\frac{\partial}{\partial X} + \frac{\partial x}{\partial x_2}\frac{\partial}{\partial x} = \frac{m_2}{M}\frac{\partial}{\partial X} - \frac{\partial}{\partial x} \;\rightarrow\; \nabla_2 = \frac{m_2}{M}\nabla_{\vec{R}} - \nabla_{\vec{r}}$$

$$\hat{T} = \frac{-\hbar^2}{2m_1}\nabla_1^2 + \frac{-\hbar^2}{2m_2}\nabla_2^2 = \frac{-\hbar^2}{2m_1}\left(\frac{m_1}{M}\nabla_{\vec{R}} + \nabla_{\vec{r}}\right)^2 + \frac{-\hbar^2}{2m_2}\left(\frac{m_2}{M}\nabla_{\vec{R}} - \nabla_{\vec{r}}\right)^2$$

$$= \frac{-\hbar^2}{2M}\nabla_{\vec{R}}^2 - \frac{\hbar^2}{2\mu}\nabla_{\vec{r}}^2$$

cross terms cancel, $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$

if $V(\vec{r}_1, \vec{r}_2, t) = V(\vec{r})$ (central force)

then let $\Psi(\vec{r}_1, \vec{r}_2, t) = \psi_R(\vec{R})\,\psi_r(\vec{r})\,e^{-iEt/\hbar}$

$$\hat{T}_{\vec{R}}\,\psi_R(\vec{R}) = E_R\,\psi_R(\vec{R}) \qquad \left[\hat{T}_r + V(\vec{r})\right]\psi(\vec{r}) = E_r\,\psi(\vec{r}) \qquad E = E_R + E_r$$

free particle. $\qquad$ single-particle, central potential, reduced mass