L66-Examples: B.E.C. and Black Body

Monday, February 29, 2016 09:29

* Summary:

$$n(\varepsilon) = \left[e^{(\mu-\varepsilon)/kT} + \begin{matrix} 1 & \text{Fermi-Dirac} \\ 0 & \text{Maxwell-Boltzmann} \\ -1 & \text{Bose-Einstein} \end{matrix}\right]^{-1}, \quad N(\varepsilon) = d(\varepsilon)\cdot n(\varepsilon)$$

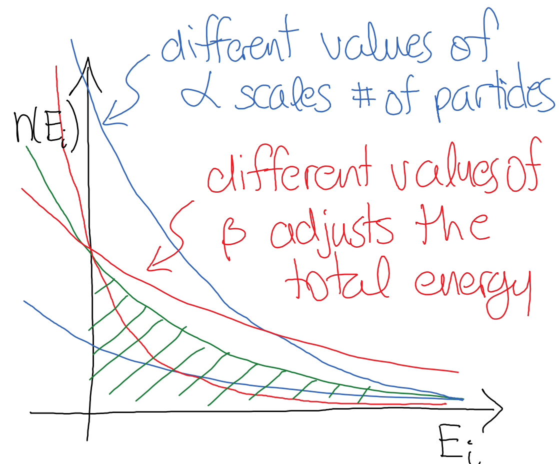
α or μ shifts distribution to conserve # of particles $N = \sum_\varepsilon d_\varepsilon n_\varepsilon$

β or T stretches distribution to accomodate energy $E = \sum_\varepsilon d_\varepsilon n_\varepsilon \cdot \varepsilon$

- Maxwell-Boltzmann distribution:

$$e^{(\mu-\varepsilon)/kT} = e^{\mu/kT}\, e^{-\varepsilon/kT}$$

The absolute activity $z = e^{-\alpha} = e^{\mu/kT}$ acts like a normalization const. We integrated to find μ last class.

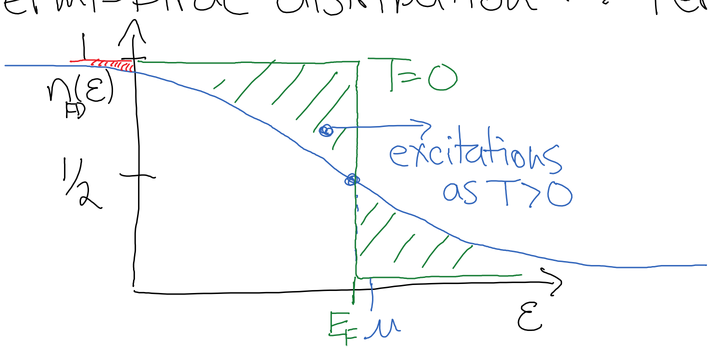


- for F.-D. (+) or B.-E. (−) distributions, α, β more difficult do integrate:

$$N = \int d_k\, n(k) = \frac{V}{2\pi^2}\int_0^\infty \frac{k^2\,dk}{e^{(\hbar^2k^2/2m-\mu)/kT} \pm 1} \Rightarrow \text{normalization } e^{-\mu/kT}$$

$$E = \int d_k\, n(k)\,\varepsilon(k) = \frac{V}{2\pi^2}\frac{\hbar^2}{2m}\int_0^\infty \frac{k^4\,dk}{e^{(\hbar^2k^2/2m-\mu)/kT} \pm 1} \Rightarrow C_V = \frac{dE}{dT} \text{ heat capacity.}$$

- Fermi-Dirac distribution → Fermi gas as $T \to 0$



note shift in $\mu(\varepsilon)$ to accomodate lost states ( ) at $\varepsilon < 0$

- Bose-Einstein distribution

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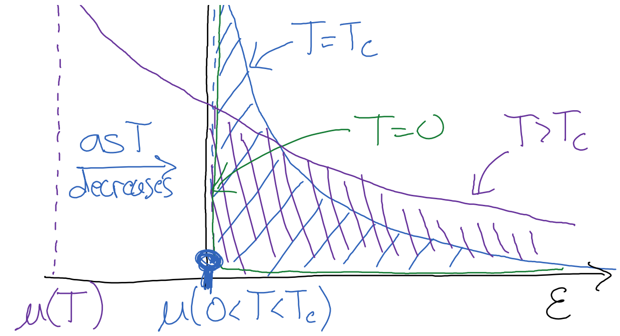


$\mu(T)<0$ at high temp. As $T\to 0$, exponential contracts, and μ increases to accomodate new particles. But $\mu<0$ so when $\mu(T_c)=0$, the maximum # of particles can fit in the distribution. The rest condense into the ground state. (A finite fraction of all particles) This is called the "Bose-Einstein Condensate".

examples: superfluid helium, ultra cold bosonic atoms

* Example: Blackbody Radiation

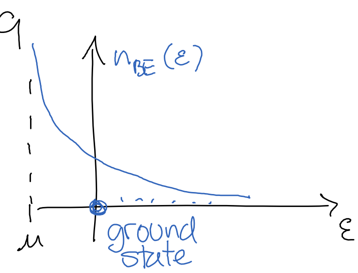
assumptions: a) $E=\hbar\omega$ b) $c=\frac{\omega}{k}$ c) $d_s=2$ left/right circ. pol

d) N is not conserved, but $\mu=0$ constant instead

then $d_k=\frac{2V}{2\pi^2}k^2dk=\frac{V}{\pi^2c^3}d\omega$ $dN_k=d_ke^{-\beta E}$

$$\rho(\omega)=\frac{dN_kE_k}{Vd\omega}=\frac{\hbar\omega}{\pi^2c^3(e^{\hbar\omega/kT}-1)}$$ as discussed last semester

* Example: Bose-Einstein condensate. #5.29



$$n(\varepsilon)=\left[e^{(\varepsilon-\mu)/kT}-1\right]^{-1}>0 \Rightarrow \frac{\varepsilon-\mu}{kT}>0$$

$$N=\frac{V}{2\pi^2}\int_0^\infty\frac{k^2dk}{e^{(\hbar^2k^2/2m-\mu)/kT}-1}=\text{constant}$$

As T decreases μ must increase to compensate for less area under the curve.

However, the maximum value of μ is at $\varepsilon = 0$ (ground state). at which point all the extra particles which cannot fit in the distribution condense into the ground state.

let $x = \frac{E}{kT} = \frac{\hbar^2 k^2}{2mkT}$ $\quad k = \frac{\sqrt{2mkTx}}{\hbar}$ $\quad dk = \frac{\sqrt{2mkT}}{2\hbar\sqrt{x}}\,dx$

at $\mu = 0$, $\frac{N}{V} = \frac{1}{2\pi^2}\left(\frac{2mkT}{\hbar^2}\right)^{3/2} \cdot \frac{1}{2}\underbrace{\int_0^\infty \frac{x^{1/2}}{e^x - 1}\,dx}_{\Gamma(3/2) = \sqrt{\pi}/2 \quad \zeta(3/2) = 2.61} = 2.61\underbrace{\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}}_{n_c}$

$n/n_c = \zeta(3/2)$ $\quad T_c = \frac{2\pi\hbar^2}{mk}\left(\frac{n}{\zeta(3/2)}\right)^{2/3}$