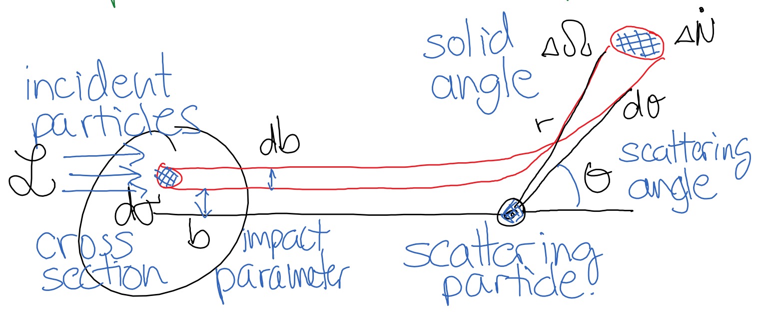
L80-Cross Sections

Wednesday, April 13, 2016 09:52

* Classical scattering cross sections §11.1.1

"Differential cross sections $d\sigma \to \frac{d\sigma}{d\Omega}$ are the meeting point between experimentalists and theorists."



$$\Delta\dot{N} = \underbrace{\mathcal{L}}_{\text{beam}}\, \frac{d\sigma}{d\Omega} \cdot \underbrace{\Delta\Omega}_{\text{detector}} \quad \text{[expt.]}$$

$$\frac{d\sigma}{d\Omega} = \frac{2\pi}{2\pi} \cdot \frac{b}{\sin\theta} \cdot \underbrace{\frac{db}{d\theta}}_{\text{physics}} \quad \text{[theory]}$$

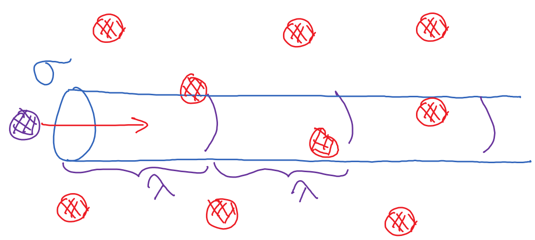
$$\mathcal{L} = \underbrace{\frac{\#\text{projectiles}}{\text{time}}}_{I_{beam}} \cdot \underbrace{\frac{\#\text{scatterers}}{\text{area}}}_{I_{targ}} = \underbrace{\#\text{projectiles}}_{N} \cdot \underbrace{\frac{\#\text{scatterers}}{\text{volume}}}_{n} \cdot \underbrace{\text{relative velocity}}_{v_{rel}} \quad \text{[luminosity]}$$

$$\Delta\Omega = \text{detector area}/r^2 = \int_{det} \underbrace{\sin\theta\, d\theta\, d\phi}_{d\Omega} \leq 4\pi \quad \text{[solid angle]}$$

$$\underbrace{\sigma}_{\text{total cross section}} = \frac{d\dot{N}}{\mathcal{L}} = \int d\sigma = \int \frac{d\sigma}{d\Omega}\, d\Omega \quad \text{where} \quad \underbrace{\frac{d\sigma}{d\Omega}}_{\text{differential cross section}} = \frac{d\dot{N}}{\mathcal{L}\,\Delta\Omega}$$

$$\underbrace{\dot{N}/V}_{\text{reaction rate}} = \frac{\#\text{interactions}}{\text{time}\cdot\text{volume}} = \underbrace{n^2}_{\text{density}} \underbrace{\langle \sigma v_{rel} \rangle}_{\text{interaction volume rate}} = \left(\frac{\#\text{atoms}}{\text{volume}}\right)^2 \left(\text{area} \cdot \frac{\text{length}}{\text{time}}\right)$$

$$n\underbrace{\lambda}_{\text{mean free path}}\sigma \equiv 1 = \frac{\#\text{atoms}}{\text{volume}} \cdot \text{length per interaction} \cdot \text{area of interation}$$

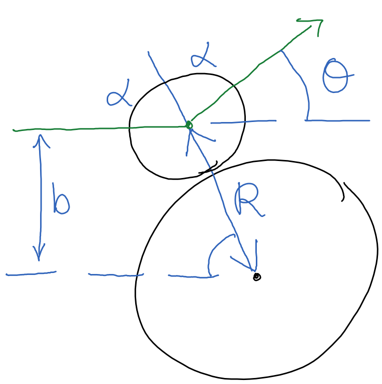


$$N = N_0 e^{-nl\sigma} = N_0 e^{-l/\lambda} \quad \text{[attenuation]}$$

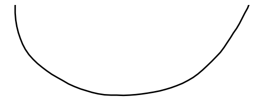
* Example: hard sphere scattering

$$b = R\sin\alpha = R\sin\left(\frac{\pi-\theta}{2}\right) = R\cos\left(\frac{\theta}{2}\right)$$

$$\frac{d\sigma}{d\Omega} = \left|\frac{b\,db}{\sin\theta\, d\theta}\right| = \left|\frac{R\cos(\theta/2)\cdot\frac{-1}{2}R\sin(\theta/2)\,d\theta}{2\cdot 2\sin(\theta/2)\cos(\theta/2)\,d\theta}\right| = \frac{1}{4}R^2$$



$$\frac{d\sigma}{d\Omega} = \left|\frac{b\,db}{\sin\theta\, d\theta}\right| = \left|\frac{R\cos(\theta/2)\cdot\frac{1}{2}R\sin(\theta/2)\,d\theta}{2\cdot 2\sin(\theta/2)\cos(\theta/2)\,d\theta}\right| = \frac{1}{4}R^2$$



$$\sigma = \int d\sigma = \int_{4\pi} \frac{1}{4}R^2 d\Omega = \pi R^2$$ geometric interaction area! "short range force"

Note: Coulomb interaction is long-range: $\sigma \to \infty$!

* Quantum mechanical scattering cross section §11.1.2

- Exactly the same concept, but in terms of probability amplitude, using probability currents instead of fluxes
- wave mechanics obscures plain geometric interpretation
- must be defined by $\frac{d\sigma}{d\Omega} \equiv \frac{\Delta \dot{N}}{\mathcal{L}\Delta\Omega}$ instead of $\Theta(b)$
- exact same formalism as 1-d transmission/reflection: solve Schrödinger equation for asymptotic outgoing wave $\left[\frac{e^{ikr}}{r}\right]$ with incoming wave $[e^{ikz}]$ as external boundary condition

$$\psi(r,\theta) \approx A\left\{\underbrace{e^{ikz}}_{\text{incoming}} + \underbrace{f(\theta)}_{\text{scattering amplitude}} \underbrace{\frac{e^{ikr}}{r}}_{\text{outgoing}}\right\}$$

$p = \hbar k$

$E = \frac{\hbar^2 k^2}{2m}$

[spherically symmetric scattering potential]

$$\frac{d\sigma}{d\Omega} = \frac{dP_{scat}/d\Omega}{dP_{inc}/da} = \frac{|\psi_{scat}|^2 dV/d\Omega}{|\psi_{inc}|^2 dV/da} = \frac{|A\, f(\theta)/r|^2 r^2 (v\,dt)\, d\Omega/d\Omega}{|A|^2\ (v\,dt)\ d\sigma/d\sigma} = |f(\theta)|^2$$

- scattering amplitude $f(180°) \approx B/A$ [reflection] $f(0°) \approx \frac{F}{A}$ [transmission] plus all the angles in between!
- generalization: S-matrix contains transition amplitudes from one asymptotic quantum state $[n\ell m]$ to another $[n'\ell'm']$
- in order to solve for $f(\theta)$, must write e^{ikz} and $\frac{e^{ikr}}{r}$ in the same basis $\to$ spherical waves: $j_\ell(kr)\, Y_{\ell m}(\Omega)$ components of $f(\theta)$ in this basis are "partial wave amplitudes" a_ℓ

in the same basis → spherical waves: $j_\ell(kr)\,Y_{\ell m}(\Omega)$
components of $f(\theta)$ in this basis are "partial wave amplitudes" a_ℓ
and can be parametrized as "phase shifts" δ_ℓ
at low energy, where $k\ell \ll 1$ so $\ell = 0$ dominates

- alternatively, at higher energy, $E \gg V$, we can perturbatively expand Schrödinger's equation, converting it to integral form using Green's functions. This expansion is particularly simple in the Born Approximation, and each term has a corresponding Feynman Diagram.