E1-Midterm solution

Thursday, March 2, 2017 16:02

[20 pts] 1. a) What are the physical implications of the commutation relations $\vec{L} \times \vec{L} = i\hbar\vec{L}$?

b) Calculate $[L_+, L_-]$ using $[L_i, L_j] = \epsilon_{ijk} i\hbar L_k$, where $L_\pm \equiv L_x \pm iL_y$. What does this imply?

c) Using $[L_z, L_\pm] = \pm\hbar L_\pm$, show that if $|\ell m\rangle$ is an eigenvector of L_z with eigenvalue $\hbar m$, then the state $L_-|\ell m\rangle$ is also an eigenvector of L_z, but with eigenvalue $\hbar(m-1)$.

d) What is the state $L_- Y_{\ell\ell}(\theta, \phi)$, disregarding normalization?

a) By the uncertainty principle, $\Delta L_x \cdot \Delta L_y \geqslant \frac{\hbar}{2} L_z$, so we cannot simultaneously measure L_x, L_y, L_z. However, since $L^2 = L_x^2 + L_y^2 + L_z^2$ does commute with all three, L^2, L_z form a complete set of commuting observables (CSCO). Thus ℓ, m charactize the stats.

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b) $[L_+, L_-] = [L_x + iL_y, L_x - iL_y]$

$$= \overset{0}{\cancel{[L_x, L_x]}} - i[L_x, L_y] + i[L_y, L_x] + \overset{0}{\cancel{[L_y, L_y]}} = 2\hbar L_z$$

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This implies that $L_x^2 + L_y^2$ doesn't factor into L_+L_-, but rather $L_x^2 + L_y^2 = L_+L_- - \hbar L_z$

c) $L_z L_- - L_- L_z = [L_z, L_-] = -\hbar L$

$$L_z [L_- |\ell m\rangle] = (L_- L_z - \hbar \quad _-)|\ell m\rangle$$

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$$= (L_- \hbar m - \hbar L_-)|\ell m\rangle = (\hbar(m-1))[L_- |\ell m\rangle]$$

Thus $L_- |\ell m\rangle \propto |\ell (m-1)\rangle$ with eigenvalue $\hbar(m-1)$.

d) $L_- |\ell \ell\rangle \propto |\ell\, \ell-1\rangle$

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[20 pts] 2. a) Compare and contrast orbital ($\vec{\boldsymbol{L}}$) and spin ($\vec{\boldsymbol{S}}$) angular momentum.

b) Calculate $\langle S_y \rangle$ for the spinor state $\chi = \begin{pmatrix} \cos\frac{\theta}{2}e^{-i\phi/2} \\ \sin\frac{\theta}{2}e^{i\phi/2} \end{pmatrix}$. Describe the other two degrees of freedom (besides θ and ϕ) of a complex two-component spinnor.

c) Given the state χ from part b), calculate the probability of measuring spin up in z?

d) Given a particle in the state χ from part b), calculate the probability of measuring $S_x = \frac{\hbar}{2}$ (including determination of the eigenvectors). If such a measurement occured, what new state would the particle be in?

a) Both are angular momentum operators with the same symmetry: $L \times L = i\hbar L$ or $S \times S = i\hbar S$

Both generate rotations, in fact you need each to rotate a specific part ($\psi(\vec{r})$ or χ_{m_s}) of the wave function.

Both generate magnetic moments, but the Landé g-factor differs by a factor of 2.

Both have (j, m) quantum numbers, where $j \in (0, \frac{1}{2}, 1, \frac{3}{2}, \dots)$ and $m = -j, -j+1, \dots j-1, j$

L is a function of spatial coordinate, S acts on

5 spinor components, but both can be represented as matrices (any QM operator can!)

L corresponds to momentum of the wave packet, S to topologically invariant (intrinsic) momentum associated with the components of the spinor.

L can have any positive integer value: 0, 1, 2, 3, ... and has an associated wave: $Y_{lm}(\theta, \phi)$ in space.

S can have any half-integral value: $0, \frac{1}{2}, 1, \frac{3}{2}, \dots$ but only one value (a property of the particle, like mass)

The both add together to form $J = L + S$, the total angular momentum, which is conserved.

Only spin determines a particle's statistics: boson ($s = 0$ or 1 or $2 \dots$) or fermion ($s = \frac{1}{2}$ or $\frac{3}{2}$ or $\frac{5}{2} \dots$).

b) $\langle S_y \rangle = \frac{\hbar}{2} \cdot \left(C_{\theta/2} e^{i\phi/2} \;\; S_{\theta/2} e^{-i\phi/2} \right) \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} C_{\theta/2} e^{-i\phi/2} \\ S_{\theta/2} e^{i\phi/2} \end{pmatrix}$

5 $= \frac{\hbar}{2} \cdot -i \cdot C_{\theta/2} S_{\theta/2} e^{i\phi} + i \cdot S_{\theta/2} C_{\theta/2} e^{-i\phi} = 2 C_{\theta/2} S_{\theta/2} \frac{e^{i\phi} - e^{-i\phi}}{2i}$

$= \frac{\hbar}{2} \cdot \sin\theta \sin\phi$

normalization $|a|^2+|b|^2=1$; global phase $e^{i\alpha}\begin{pmatrix}a\\b\end{pmatrix}$

c) $P(S_z=\hbar/2) = \left|(1\ 0)\begin{pmatrix}C_{\theta/2}e^{-i\phi/2}\\ S_{\theta/2}e^{i\phi/2}\end{pmatrix}\right|^2 = \cos^2\theta/2$

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d) $\sigma_x \chi = \lambda \chi$ $\quad \begin{vmatrix}0-\lambda & 1\\ 1 & 0-\lambda\end{vmatrix} = \lambda^2-1=0 \quad \lambda=\pm 1$

$(\sigma_x-\lambda I)\chi=0$

$\lambda=1$: $\begin{pmatrix}-1&1\\1&-1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \quad \chi_+^x=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$

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$P(S_x=\hbar/2) = \left|\frac{1}{\sqrt{2}}(1\ \ 1)\begin{pmatrix}C_{\theta/2}e^{-i\phi/2}\\ S_{\theta/2}e^{i\phi/2}\end{pmatrix}\right|^2 = \frac{1}{2}\left|C_{\theta/2}e^{-i\phi/2}+S_{\theta/2}e^{i\phi/2}\right|^2$

3 $\quad = \frac{1}{2}\left(C_{\theta/2}e^{i\phi/2}+S_{\theta/2}e^{-i\phi/2}\right)\left(C_{\theta/2}e^{-i\phi/2}+S_{\theta/2}e^{i\phi/2}\right)$

$= \frac{1}{2}\left(C^2_{\theta/2}+S^2_{\theta/2}+C_{\theta/2}S_{\theta/2}\left(e^{i\phi/2}+e^{-i\phi/2}\right)\right) = \frac{1}{2}(1+S_\theta C_\phi)$

After positive measurement the state would be $\chi_+^x=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$.

[20 pts] 3. a) Explain the physical origin of the band-like structure of energy levels in a periodic potential such as a crystal.

b) Calculate the maximum momentum p_F of any single particle in the ground state of a system of N fermions confined in an 2-dimensional infinite square well $0 < x < L$ and $0 < y < L$. [-5 pts for soving the problem in 1-d instead of 2-d]

a) It comes from degeneracies in the wavefunctions of a potential with localized wells separated from each other. Between each well, the wavefunction can cross over zero, or go back in the same direction without much change in curvature (shape)

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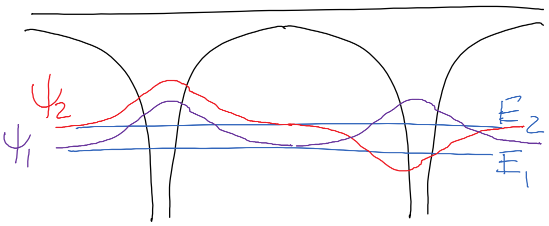

b) $\left[\frac{-\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+\cancelto{0}{V(x)}-E\right]\psi(x,y)=0$ $\quad$ let $\psi(x,y)=\sin(k_xx)\sin(k_yy)$

5 $\quad \left[\frac{-\hbar^2}{2m}\left(-k_x^2+-k_y^2\right)-E\right]\sin(k_xx)\sin(k_yy)=0 \quad E=\frac{\hbar^2}{2m}(k_x^2+k_y^2)$

Boundary conditions: $\psi(x,0) = \psi(y,0) = 0$ satisfied by sin.

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$$\psi(L,y) = \sin(k_x L)\cdot\sin(k_y y) = 0 \Rightarrow k_x L = n_x \pi$$
$$\psi(x,L) = \sin(k_x x)\cdot\sin(k_y \cdot L) = 0 \Rightarrow k_y L = n_y \pi$$

$$N = \frac{1}{4}\pi n^2 = \frac{\pi}{4}(n_x^2 + n_y^2)\cdot 2$$

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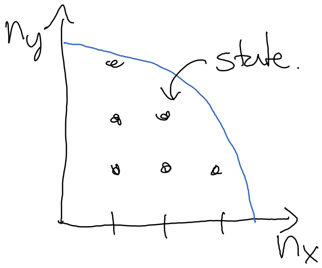
$$= \frac{\pi}{2}\left(\left(\frac{k_x L}{\pi}\right)^2 + \left(\frac{k_y L}{\pi}\right)^2\right) = \frac{L^2}{2\pi} k_F^2$$

thus $k_F = \frac{\sqrt{2\pi N}}{L}$



1-d: $\psi(x) = \sin(k_x L)$ $\quad k_x L = n\pi$

for N particles, the highest state is N/2

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thus $k_F = \frac{\pi N}{2L}$

[20 pts] 4. A quantum system has $N = 4$ particles which can occupy three single-particle energy levels $E_1 = 0$ eV, $E_2 = 1$ eV, $E_3 = 2$ eV with degeneracy $d_1 = 1, d_2 = 2, d_3 = 3$, respectively.

a) How many microstates are there in the configuration $N_1 = 1$, $N_2 = 1$, $N_3 = 2$ of distinguishable particles?

b) Draw each of the microstates of 4 fermions in the configuration of part a).

c) What configurations are possible for 4 bosons in this system with total energy $E = 4$ eV?

d) Which configuration of part c) is most probable?

a) The particle in E_1 has 1 choice.
The particle in E_2 has 2 choices.
The 2 particles in E_3 has $3^2 = 9$ choices.
There are $\binom{4}{1\;1\;2} = \frac{4!}{1!1!2!} = \frac{24}{2} = 12$
ways of binning the 4 particles into energy levels $\rightarrow$ $Q = 1\cdot 2\cdot 9\cdot 12 = 216$ microstates.

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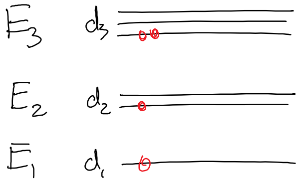

b)

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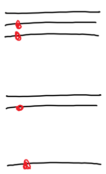
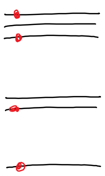
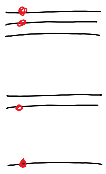
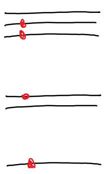
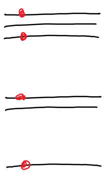
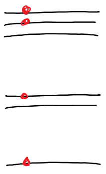




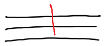








c)



E=2

3 configurations: $(N_1, N_2, N_3) =$

c)

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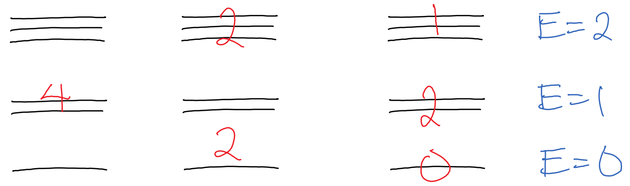

3 configurations: $(N_1, N_2, N_3) =$
i) $(0, 4, 0)$
ii) $(2, 0, 2)$
iii) $(0, 2, 1)$

d) $Q = \prod_n \binom{N_n + d_n - 1}{N_n}$ $=$ i) $1 \cdot \binom{4+2-1}{4} \cdot 1 = 5$

5 ii) $\binom{2+0}{2} \cdot 1 \cdot \binom{2+3-1}{2} = 1 \cdot 1 \cdot 6 = 6$

iii) $1 \cdot \binom{2+2-1}{2} \cdot \binom{1+3-1}{1} = 1 \cdot 3 \cdot 3 = 9$ ← the most probable.

[20 pts] 5. a) Explain the meaning of the Lagrange multipliers α and β, used to calculate the statistical distribution of multiparticle systems, and the effect they each have on the distributions of distiniguishable particles, bosons, or fermions.

Using the method of Lagrange multipliers, minimize the function $Q = N_1^2 + N_2^2 + N_3^2$ subject to the constraints $N_1 + N_2 + N_3 = N$ and $N_1 + 2N_2 + 3N_3 = E$, for $N = 6$ and $E = 14$. Unlike the case studied in class, there is no advantage in using $\log(Q)$ instead of Q in this problem. [−5 points for using only one instead of both constraints]

a) $\alpha = \frac{-\mu}{kT}$ μ is the chemical potential, the energy associated with each particle in the system

6 increasing α increases the total # of particles N

$\beta = \frac{1}{kT}$ T is the temperature, the average of each particle in the system

decreasing β increases the total thermal energy E

b) $G = Q + \alpha(N - \sum_n N_n) + \beta(E - \sum_n N_n E_n)$

4 $= N_1^2 + N_2^2 + N_3^2 + \alpha(N - (N_1 + N_2 + N_3)) + \beta(E - (1N_1 + 2N_2 + 3N_3))$

4 $\frac{\partial G}{\partial N_n} = 2N_n - \alpha - E_n \beta = 0$ $N_n = \frac{1}{2}(\alpha + E_n \beta)$

$N = \sum_n N_n = \sum_n \frac{1}{2}(\alpha + E_n \beta) = \frac{3}{2}\alpha + \frac{1}{2} \sum_n E_n \beta$

2 $6 = N_1 + N_2 + N_3 = \frac{1}{2}(\alpha + E_1 \beta) + \frac{1}{2}(\alpha + E_2 \beta) + \frac{1}{2}(\alpha + E_3 \beta) = \frac{3}{2}\alpha + \frac{6}{2}\beta$

$$E = \sum_n N_n E_n = \sum_n \frac{1}{2}(\alpha + E_n \beta) E_n = \frac{1}{2} \sum_n E_n \alpha + \frac{3}{2} \sum_n E_n^2 \beta$$

2

$$14 = N_1 E_1 + N_2 E_2 + N_3 E_3 = \frac{6}{2}\alpha + \frac{14}{2}\beta = 3\alpha + 7\beta$$

2 $\alpha = 0, \ \beta = 2 \qquad N_1 = 1 \qquad N_2 = 2 \qquad N_3 = 3$

check: $N_1 + N_2 + N_3 = 6 \qquad N_1 E_1 + N_2 E_2 + N_3 E_3 = 1 + 4 + 9 = 14$