E1-Midterm solution

Thursday, March 2, 2017 16:02

[20 pts] 1. a) What are the physical implications of the commutation relations $\vec{L} \times \vec{L} = i\hbar\vec{L}$?

b) Calculate $[L_z, L_-]$, where $L_\pm \equiv L_x \pm iL_y$, and $[L_i, L_j] = \epsilon_{ijk} i\hbar L_k$.

c) Derive, using part b), that if $L_z|\ell m\rangle = \hbar m|\ell m\rangle$, then the state $L_-|\ell m\rangle$ is also an eigenvector of L_z, and calculate its eigenvalue.

d) What is the state $L_- Y_{\ell,-\ell}(\theta,\phi)$, disregarding normalization?

a) By the uncertainty principle, $\Delta L_x \cdot \Delta L_y \geq \frac{\hbar}{2} L_z$, so we cannot simultaneously measure L_x, L_y, L_z.
However, since $L^2 = L_x^2 + L_y^2 + L_z^2$ does commute with all three, L^2, L_z form a complete set of commuting observables (CSCO). Thus ℓ, m charactize the stats.
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It also it also implies the ladder algebra w/ $L_\pm = L_x \pm i L_y$ and thus the compete structure of eigenstates $|\ell m\rangle$.

b) $[L_z, L_-] = [L_z, L_x - iL_y]$

6 $= [L_z, L_x] - i[L_z, L_y] = i\hbar L_y - i(-i\hbar L_x) = -\hbar(L_x - iL_y) = -\hbar L_-$

c) $L_z L_- - L_- L_z = [L_z, L_-] = -\hbar L$

$L_z [L_- |\ell m\rangle] = (L_- L_z - \hbar L_-)|\ell m\rangle$

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$= (L_- \hbar m - \hbar L_-)|\ell m\rangle = (\hbar(m-1))[L_- |\ell m\rangle]$

Thus $L_- |\ell m\rangle \propto |\ell (m-1)\rangle$ with eigenvalue $\hbar(m-1)$.

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d) $L_- |\ell -\ell\rangle = 0$

[20 pts] 2. a) Compare and contrast orbital ($\vec{L}$) and spin ($\vec{S}$) angular momentum.

b) Show that the spinor $\chi = \begin{pmatrix} \cos\frac{\theta}{2} e^{-i\phi/2} \\ \sin\frac{\theta}{2} e^{i\phi/2} \end{pmatrix}$ is an eigenvalue of the operator $\vec{S} \cdot \hat{n}$, where the normal $\hat{n} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$ points in the (θ, ϕ) direction in spherical coordinates. Show that χ is normalized.

c) Calculate the time evolution of the state χ from part b) in a magnetic field $\vec{B} = \hat{n}B$.

d) If the component of spin along the x-axis is measured, calculate the probability of measuring spin up and the probability of measuring spin down. Would the new state be after the measurement in each case? You may use the z-axis for partial credit.

a) Both are angular momentum operators with the same symmetry: $L \times L = i\hbar L$ or $S \times S = i\hbar S$
Both generate rotations, in fact you need each to rotate a specific part ($\psi(\vec{r})$ or χ_{m_s}) of the wave function.
Both generate magnetic moments, but the Landé g-factor differs by a factor of 2.
Both have (j, m) quantum numbers, where $j \in (0, \frac{1}{2}, 1, \frac{3}{2}, \ldots)$ and $m = -j, -j+1, \ldots j-1, j$
L is a function of spatial coordinate, S acts on
5 spinor components, but both can be represented as matrices (any QM operator can!)
L corresponds to momentum of the wave packet, S to topologicaly invariant (intrinsic) momentum associated with the components of the spinor.
L can have any positive integer value: $0, 1, 2, 3, \ldots$ and has an associated wave: $Y_{\ell m}(\theta, \phi)$ in space.
S can have any half-integal value: $0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$ but only one value (a property of the particle, like mass)
The both add together to form $J = L + S$, the total angular momentum, which is conserved.
Only spin determines a particle's statistics: boson ($s = 0$ or 1 or $2 \ldots$) or fermion ($s = \frac{1}{2}$ or $\frac{3}{2}$ or $\frac{5}{2} \ldots$).

b) $\vec{S} \cdot \hat{n} = \frac{\hbar}{2} \left(\sigma_x S_\theta C_\phi + \sigma_y S_\theta S_\phi + \sigma_z C_\theta \right) =$

$C_\theta = C_{\theta/2}^2 - S_{\theta/2}^2$
$S_\theta = 2 C_{\theta/2} S_{\theta/2}$

5 $\vec{S} \cdot \hat{n}\, \chi = \frac{\hbar}{2} \begin{pmatrix} C_\theta & S_\theta e^{-i\phi} \\ \ldots \end{pmatrix} \begin{pmatrix} C_{\theta/2} e^{-i\phi/2} \\ \ldots \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} C_\theta C_{\theta/2} e^{-i\phi/2} + S_\theta S_{\theta/2} e^{-i\phi/2} \\ \ldots \end{pmatrix}$

5 $\vec{S}\cdot\hat{n}\,\chi = \frac{\hbar}{2}\begin{pmatrix} c_\theta & s_\theta e^{-i\phi} \\ s_\theta e^{i\phi} & -c_\theta \end{pmatrix}\begin{pmatrix} c_{\theta/2}e^{-i\phi/2} \\ s_{\theta/2}e^{i\phi/2} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} c_\theta c_{\theta/2}e^{-i\phi/2} + s_\theta s_{\theta/2}e^{-i\phi/2} \\ s_\theta c_{\theta/2}e^{i\phi/2} - c_\theta s_{\theta/2}e^{i\phi/2} \end{pmatrix}$

$= \frac{\hbar}{2}\begin{pmatrix} [(c^2_{\theta/2} - s^2_{\theta/2})c_{\theta/2} + 2c_{\theta/2}s_{\theta/2}s_{\theta/2}]e^{-i\phi/2} \\ [(2c_{\theta/2}s_{\theta/2})c_{\theta/2} - (c^2_{\theta/2} - s^2_{\theta/2})s_{\theta/2}]e^{i\phi/2} \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} (\overbrace{c^2_{\theta/2} + s^2_{\theta/2}}^{1})c_{\theta/2}e^{-i\phi/2} \\ (\overbrace{c^2_{\theta/2} + s^2_{\theta/2}}^{1})s_{\theta/2}e^{i\phi/2} \end{pmatrix} = \frac{\hbar}{2}\chi$

normalization: $\chi^\dagger\chi = (c_{\theta/2}e^{+i\phi/2}\ \ s_{\theta/2}e^{-i\phi/2})\begin{pmatrix} c_{\theta/2}e^{-i\phi/2} \\ s_{\theta/2}e^{i\phi/2} \end{pmatrix} = c^2_{\theta/2} + s^2_{\theta/2} = 1$

c) $H = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{S}\cdot\hat{n}B$ $\quad H\chi = -\gamma\vec{S}\cdot\hat{n}B\chi = -\gamma B\frac{\hbar}{2}\chi = E\chi$

5 $\chi(t) = \chi e^{-iEt/\hbar} = \chi e^{-i(-\gamma B\hbar/2)t/\hbar} = \begin{pmatrix} c_{\theta/2}e^{-i\phi/2} \\ s_{\theta/2}e^{i\phi/2} \end{pmatrix}e^{i\gamma B t/2}$ (stationary state)

d) $\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $\quad P_+ = |\chi_+^{(x)\dagger}\chi|^2 = \frac{1}{2}|c_{\theta/2}e^{-i\phi/2} + s_{\theta/2}e^{i\phi/2}|^2 = \frac{1}{2}[c^2_{\theta/2} + s^2_{\theta/2} + 2c_{\theta/2}s_{\theta/2}\mathrm{Re}\,e^{i\phi}]$

5 $= \frac{1}{2}(1 + s_\theta c_\phi)$. New state would be $\to \chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad P_- = |\chi_-^{(x)}\chi|^2 = \frac{1}{2}|c_{\theta/2}e^{-i\phi/2} - s_{\theta/2}e^{i\phi/2}|^2 = \frac{1}{2}(1 - s_\theta c_\phi) \to \chi_-^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

[20 pts] 3. a) Write the full antisymmetric wave function $\psi(\vec{r}_1, \vec{r}_2)\chi(s, m_s)$ of the ground state a two-electron atom in terms of its single-particle wave functions $\psi_{n\ell m_\ell}(\vec{r})\chi_{m_s}$, ignoring the interaction between the two electrons. Identify the particle exchange symmetry of both the spatial (ψ) and spinor (χ) parts of the wave function.

b) Show that a 1-dimensional potential $V(x) = V(x + a)$ with period a commutes with the translation operator D, where $D\psi(x) = \psi(x + a)$. Given periodic boundary conditions $\psi(Na) = \psi(0)$ and $\psi'(Na) = \psi'(0)$, (for example, a crystal with N atoms around a ring), show that the wave function $\psi(x)$ has the periodic structure $\psi(x + a) = e^{iK_n a}\psi(x)$, where $K_n = 2\pi n/Na$ for $n = 0, 1, 2, \ldots N-1$. What is the physical interpretation of K_n and how is it manifest in the energy spectrum of the crystal?

a) Coupling the two spins, $\chi(s, m_s) = |1, m\rangle = \begin{cases} \uparrow\uparrow \\ \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow) \\ \downarrow\downarrow \end{cases}$ symmetric triplet

5 or $|0,0\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$ antisymmetric singlet.

The spatial part must then be either

5 $\psi(\vec{r}_1, \vec{r}_2) = \psi_{100}(\vec{r}_1)\cdot\psi_{100}(\vec{r}_2)$ (singlet), symmetric

because the antisymmetric combination $= 0$.

thus $\psi(\vec{r}_1, \vec{r}_2)\,\chi(s, m_s) = \psi_{100}(\vec{r}_1) \cdot \psi_{100}(\vec{r}_2) \cdot \frac{1}{\sqrt{2}}(\uparrow\downarrow - \downarrow\uparrow)$

assuming the single-particle ground state is $\overset{n\ l\ m}{(1, 0, 0)}$

b) $D V(x) \psi(x) = V(x+a)\psi(x+a) = V(x) D \psi(x) = V(x) D\, \psi(x)$

5 thus $[D, V] = 0$ and they are simultaneously diagonalizable

let $\psi(x)$ be an eigenstate of both, then

$D\psi(x) = \psi(x+a) = \lambda \psi(x)$, λ is the eigenvalue of D.

5 but $\psi(0) = \psi(Na) = \lambda^N \psi(0)$ thus $\lambda^N = 1$ $\lambda = e^{i 2\pi/N \cdot n} = e^{i K_n a}$

where $K_n = \frac{2\pi n}{Na}$ is the spacing between levels in a band

[20 pts] 4. A quantum system has $N = 4$ particles which can occupy three single-particle energy levels $E_1 = 0$ eV, $E_2 = 1$ eV, $E_3 = 2$ eV with degeneracy $d_1 = 3, d_2 = 2, d_3 = 1$, respectively.

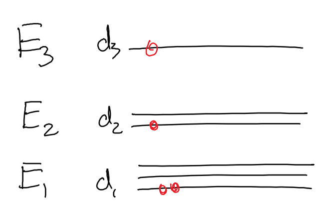
a) How many microstates are there in the configuration $N_1 = 2$, $N_2 = 1$, $N_3 = 1$ of distinguishable particles?

b) Draw each of the microstates of 4 fermions in the configuration of part a).

c) What configurations are possible for 4 bosons in this system with total energy $E = 4$ eV?

d) Which configuration of part c) is most probable?

a) The 2 particles in E_1 has $3^2 = 9$ choices.
The particle in E_2 has 2 choices.
The particle in E_3 has 1 choice.



5 There are $\binom{4}{1\ 1\ 2} = \frac{4!}{1!\,1!\,2!} = \frac{24}{2} = 12$

ways of binning the 4 particles into energy levels $\rightarrow$ $\Omega = 1 \cdot 2 \cdot 9 \cdot 12 = 216$ microstates.

b)

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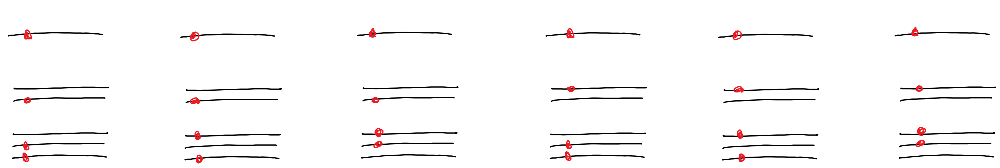

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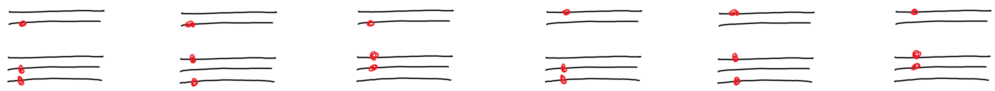
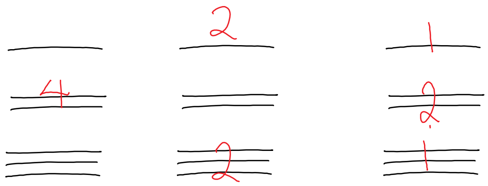


c)



$E=2$

$E=1$

$E=0$

3 configurations: $(N_1, N_2, N_3) =$

i) $(0, 4, 0)$

ii) $(2, 0, 2)$

iii) $(1, 2, 1)$

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d) $Q = \prod_n \binom{N_n + d_n - 1}{N_n} =$ i) $1 \cdot \binom{4+2-1}{4} \cdot 1 = 5$

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ii) $\binom{2+3-1}{2} \cdot 1 \cdot \binom{2+0}{2} = 6 \cdot 1 \cdot 1 = 6$

iii) $\binom{1+3-1}{1} \cdot \binom{2+2-1}{2} \cdot 1 = 3 \cdot 3 \cdot 1 = 9$ ← the most probable.