University of Kentucky, Physics 521
Homework #13, Rev. A, due Wednesday, 2018-01-24

0. Griffiths [2ed] Ch. 3 #39; Ch. 4 #27, #30, #31, #49, #52, #53.

1. Clifford algebra. The complete 3-vector algebra including dot and cross products can be implemented using the identity (I) as the unit scalar and Pauli matrices ($\boldsymbol{\sigma}$) as unit vectors:

$$1 = I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \hat{\boldsymbol{x}} = \boldsymbol{\sigma}_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \hat{\boldsymbol{y}} = \boldsymbol{\sigma}_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \hat{\boldsymbol{z}} = \boldsymbol{\sigma}_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

The dot and cross products can be represented by matrix multiplication, as in the formula

$$\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j = I(\boldsymbol{\sigma}_i \cdot \boldsymbol{\sigma}_j) + i(\boldsymbol{\sigma}_i \times \boldsymbol{\sigma}_j) = I\delta_{ij} + i\epsilon_{ijk}\boldsymbol{\sigma}_k, \tag{2}$$

where $\boldsymbol{\sigma}_{i,j}$ in the dot and cross products are interpreted as unit vectors. Note the difference between the imaginary i and the index $_i$. Also note that scalars are often implicitly multiplied by I when adding with other matrices. Also note this algebra generalizes to a space-time algebra using the Dirac matrices γ^μ instead of the Pauli matrices σ_i.

a) Verify Eq. 2 for all nine products $\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j$ and show it is equivalent to the expression

$$(\boldsymbol{\sigma} \cdot \boldsymbol{a})(\boldsymbol{\sigma} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} + i\boldsymbol{\sigma} \cdot (\boldsymbol{a} \times \boldsymbol{b}). \tag{3}$$

Here, σ_x, σ_y, and σ_z are treated as to components of a vector $\boldsymbol{\sigma}$, although a more natural interpretation is as unit vectors, so that $\boldsymbol{\sigma} \cdot \boldsymbol{a} = \sigma_x a_x + \sigma_y a_y + \sigma_z a_z \approx \hat{\boldsymbol{x}} a_x + \hat{\boldsymbol{y}} a_y + \hat{\boldsymbol{z}} a_z$.

b) Which products are symmetric and which are antisymmetric, ie. $\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j = \pm \boldsymbol{\sigma}_j \boldsymbol{\sigma}_i$? In general, a matrix product has both symmetric and antisymmetric parts, but the simple in terms of $i = j$ and $i \neq j$ defines a Clifford algebra.

c) Show that any linear product $\boldsymbol{a} \circ \boldsymbol{b}$, can be decomposed into the sum $\boldsymbol{a} \circ \boldsymbol{b} = \{\boldsymbol{a} \circ \boldsymbol{b}\} + \langle \boldsymbol{a} \circ \boldsymbol{b} \rangle$ of symmetric $\{\boldsymbol{a} \circ \boldsymbol{b}\} \equiv \frac{1}{2}(\boldsymbol{a} \circ \boldsymbol{b} + \boldsymbol{b} \circ \boldsymbol{a})$ and antisymmetric $\langle \boldsymbol{a} \circ \boldsymbol{b} \rangle \equiv \frac{1}{2}(\boldsymbol{a} \circ \boldsymbol{b} - \boldsymbol{b} \circ \boldsymbol{a})$ parts, with respect to exchange of $\boldsymbol{a}$ and $\boldsymbol{b}$. Show that $\langle \boldsymbol{a} \circ \boldsymbol{a} \rangle = 0$ always. Apply this decomposition to the product $\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j$. In the same manner, any matrix can be decomposed as the sum of a symmetric and antisymmetric matrix. Why are the diagonal elements of an antisymmetric matrix zero?

d) The imaginary i in the above formula is not present in the ordinary cross product. It distinguishes [axial] pseudovectors from [polar] vectors. Using part a), calculate the value of the pseudoscalar $\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j \boldsymbol{\sigma}_k$, where $i \neq j \neq k$, and show it is completely antisymmetric in i, j, k. What is the analog of this triple product in terms of dot and cross products?

2. Generators of rotation. In Griffiths #3.39, we showed that $p_x/\hbar$ is the generator of translation and $\mathcal{H}/\hbar$ is the generator of time evolution of the wavefunction: $\exp(-ip_x x_0/\hbar)\psi(x) = \psi(x - x_0)$ and $\exp(-i\mathcal{H}t_0/\hbar)\Psi(x,t) = \Psi(x, t + t_0)$. We saw another example in H09, where $M_z = \hat{\boldsymbol{z}}\times = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generates the orthogonal matrix $R_\phi = \exp(M_z\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$, which rotates 2-dimensional vectors (spin 1, not two-component $s=\frac{1}{2}$ spinors). In 3-d, $\boldsymbol{M} = (M_x, M_y, M_z)$ generates the rotation $R_{\boldsymbol{\omega}} = \exp(\boldsymbol{M} \cdot \boldsymbol{\omega}) = I \cos\omega + \boldsymbol{M} \cdot \hat{\boldsymbol{\omega}} \sin\omega + \hat{\boldsymbol{\omega}}\hat{\boldsymbol{\omega}}^T(1 - \cos\omega)$ of 3-vectors about the axis $\boldsymbol{\omega}$.

a) In analogy with p_x, show that L_z generates the rotation of a wave function about the z-axis: $\exp(-i\phi_0 L_z/\hbar)\psi(r, \theta, \phi) = \psi(r, \theta, \phi - \phi_0)$. This generalizes to $\exp(i\boldsymbol{L} \cdot \boldsymbol{\omega})\psi(\boldsymbol{r}) = \psi(R_{\boldsymbol{\omega}}\boldsymbol{r})$, where $R_{\boldsymbol{\omega}}$ is a normal rotation matrix for vectors.

b) Show that the generators $\boldsymbol{M}$ are the cartesian equivalent of the 3×3 spin s=1 generator matrices $\boldsymbol{iS}/\hbar$ in spherical tensor components of Griffiths #4.31. Hint: The vector $\boldsymbol{v} = (v_x, v_y, v_z)$ has spherical tensor components $v_{\pm 1} = \frac{1}{\sqrt{2}}(v_x \pm i v_y)$ and $v_0 = v_z$, which are not the same as its spherical components $\boldsymbol{v} = \hat{\boldsymbol{r}} v_r + \hat{\boldsymbol{\theta}} v_\theta + \hat{\boldsymbol{\phi}} v_\phi$.

c) Calculate the $s = \frac{1}{2}$ spinnor rotation matrices $R_i(\phi) = \exp(i\sigma_i\phi/2)$ about the $i = x, y, z$ axes. How do these relate to the eigenvectors of $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{n}}$ in Griffiths #4.30? Show that a spinnor changes sign after a full revolution and must be rotated by 4π to return back to its original value.