L52-Angular Momentum-Eigenvalues

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* Review: $\vec{L} \equiv \vec{r} \times \vec{p}$ $\qquad [p_i, r_j] = i\hbar\,\delta_{ij}$

by cyclic permutation, $\boxed{[L_i, L_j] = \varepsilon_{ijk}\, i\hbar\, L_k}$

or formally, $\boxed{\hat{L} \times \hat{L} = i\hbar\, \hat{L}}$

"generalized definition of angular momentum $\vec{J}$"
This captures all of its essential properties.
The eigenvalues can be derived from $\vec{J} \times \vec{J} = i\hbar\,\vec{J}$

* define $\hat{L}^2 \equiv L_x^2 + L_y^2 + L_z^2$ to complete the $\vec{L}$ algebra

then $[L_x, L^2] = \cancelto{0}{[L_x, L_x^2]} + [L_x, L_y^2] + [L_x, L_z]^2$

$= [L_x, L_y]L_y + L_y[L_x, L_y] + [L_x, L_z]L_z + L_z[L_x, L_z]$

$= i\hbar(L_zL_y + L_yL_z) - i\hbar(L_yL_z + L_zL_y) = 0$

thus $\boxed{[L^2, \vec{L}] = 0}$

we can choose L^2, L_z to be a complete set of commuting operators over the space of angular functions. (these appear in ∇^2!)

* can we factor $L_\perp^2 \equiv (L_x^2 + L_y^2) \stackrel{?}{=} (L_x + iL_y)(L_x - iL_y)$ NO! (commutation relations!)

define: $L_\pm \equiv L_x \pm iL_y \qquad L_\pm^\dagger = L_\mp$ ladder operators $[L_\pm, L^2] = 0$ still.

$L_+L_- = (L_x + iL_y)(L_x - iL_y) = L_x^2 + L_y^2 - i[L_x, L_y] = L_x^2 + L_y^2 + \hbar L_z$

$L_-L_+ = L_x^2 + L_y^2 - \hbar L_z$ so $[L_+, L_-] = 2[L_x, L_y] = 2\hbar L_z$

$[L_z, L_\pm] = [L_z, L_x] \pm i[L_z, L_y] = i\hbar(L_y \mp iL_x) = \pm\hbar L_\pm$

thus $L_z L_\pm = (L_\pm \pm \hbar) L_z$ it raises or lowers eigenvalues

* summary of commutation relations:

$$\begin{array}{c|cccccc}
 & \multicolumn{2}{c}{\text{CSCO}} & & B & \multicolumn{2}{c}{\text{ladders}} \\
[A,B] & L^2 & L_z & L_x & L_y & L_+ & L_- \\
\hline
L^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
L_z & 0 & 0 & i\hbar L_y & -i\hbar L_x & \multicolumn{2}{c}{\pm\hbar L_\pm} \\
A \quad L_x & 0 & -i\hbar L_y & 0 & i\hbar L_z & \multicolumn{2}{c}{\mp\hbar L_z} \\
L_y & 0 & i\hbar L_x & -i\hbar L_z & 0 & \multicolumn{2}{c}{-i\hbar L_z} \\
L_+ & 0 & & (\text{antisym}) & & 0 & 2\hbar L_z \\
L_- & 0 & & & & & 0
\end{array}$$

ladders.

factorization.

Plan to determine eigenvalues:

A) $L_\pm |\ell m\rangle = A^\pm_{\ell m} |\ell, m\pm 1\rangle$

B) $L_z |\ell m\rangle = \hbar m |\ell m\rangle$

C) $L^2 |\ell m\rangle = \hbar \ell(\ell+1) |\ell m\rangle$

D) $m = -\ell, -\ell+1, \dots \ell-1, \ell$

D) $A^\pm_{\ell m} = \hbar\sqrt{(\ell \mp m)(\ell \pm m+1)}$

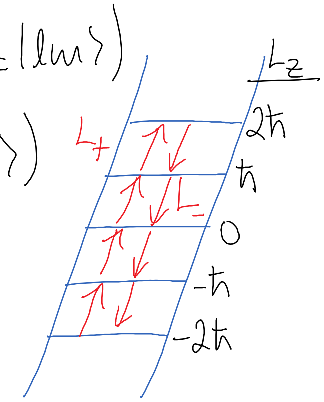
* Ladder property: let $|\ell m\rangle$ be an eigenstate of L^2, L_z

with $\hat{L}^2 |\ell m\rangle = \lambda |\ell m\rangle \qquad L_z |\ell m\rangle = \mu |\ell m\rangle$

and consider the new state $L_\pm |\ell m\rangle$

A) $L^2 (L_\pm |\ell m\rangle) = L_\pm L^2 |\ell m\rangle = L_\pm \lambda |\ell m\rangle = \lambda (L_\pm |\ell m\rangle)$

$L_z (L_\pm |\ell m\rangle) = (L_\pm \pm \hbar) L_z |\ell m\rangle = (\mu \pm \hbar)(L_z |\ell m\rangle)$



B) thus $\mu = \mu_0, \mu_0 + \hbar, \mu_0 + 2\hbar, \dots = \hbar m$

$L_z |\ell m\rangle = \hbar m |\ell m\rangle$ and $L_\pm |\ell m\rangle \propto |\ell, m\pm 1\rangle$

C) let the maximum eigenvalue of L_z be $m = \ell$ so $L_+ |\ell\ell\rangle = 0$

then $L^2 |\ell\ell\rangle = \left[(L_- \cancel{L_+} + \hbar L_z) + L_z^2\right] |\ell\ell\rangle = \underbrace{\hbar^2 \ell(\ell+1)}_{\lambda} |\ell\ell\rangle$

D) at the bottom rung, $L_- |\ell \bar{\ell}\rangle = 0$ (min eig. $m = \bar{\ell}$)

then $L^2 |\ell\bar{\ell}\rangle = \left[(L_+ \cancel{L_-} - \hbar L_z) + L_z^2\right] |\ell\bar{\ell}\rangle = \underbrace{\hbar^2 \bar{\ell}(\bar{\ell}-1)}_{\text{also } \lambda} |\ell\bar{\ell}\rangle$

thus $\ell(\ell+1) = \bar{\ell}(\bar{\ell}-1)$: $\bar{\ell} = \ell+1$ or $\bar{\ell} = -\ell$ but $\bar{\ell} < \ell$

so $-\hbar l \leq \mu \leq \hbar l$ in steps of $\hbar$, which implies

$l \in \{0, \tfrac{1}{2}, 1, 1\tfrac{1}{2}, 2, \ldots\}$; $\mu = \hbar m$ $m \in \{-l, -l+1, \ldots l-1, l\}$

that is why we index the state by [half] integers l, m.

$$\boxed{\begin{aligned} L^2 |lm\rangle &= \hbar^2 l(l+1)\,|lm\rangle \\ L_z |lm\rangle &= \hbar m\,|lm\rangle \end{aligned}}$$

In our case $|lm\rangle \sim Y_{lm}(\theta,\phi)$
$l = 0, 1, 2, \ldots$ for orbital angular momentum

E) Matrix elements of ladder operators:

recall $\left[\langle lm| L_\mp\right]^\dagger = L_\pm |lm\rangle = A^\pm_{lm} |l, m\pm 1\rangle$ $\quad L_\pm^\dagger = L_\mp$ $\quad \langle l'm'|lm\rangle = \delta_{ll'}\delta_{mm'}$

$$\langle lm| L_\mp L_\pm |lm\rangle = |A^\pm_{lm}|^2 = \langle lm| L^2 - L_z^2 \mp \hbar L_z |lm\rangle$$

$$= \hbar^2\left[l(l+1) - m(m\pm 1)\right] = \hbar^2\left[(l\mp m)(l+1\pm m)\right]$$

thus $A^\pm_{lm} = \hbar\sqrt{(l\mp m)(l\pm m+1)}$ up to an arbitrary phase

note: $A^+_{ll} = 0$ since $L_+|ll\rangle = 0$ and $A^-_{l-l} = 0$ since $L_-|l\,{-l}\rangle = 0$

so $\boxed{L_\pm |lm\rangle = \hbar\sqrt{(l\mp m)(l\pm m+1)}\,|l, m\pm 1\rangle}$