L54-Spin

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* QM spin is NOT: classical spin !

- it is not "internal" angular momentum (of constituents inside the particle)

$$\vec{J} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i (\vec{r}_{cm} + \vec{r}_i{}') \times (\vec{p}_{i\,cm} + \vec{p}_i{}')$$

$$\sum_i m_i (1, \vec{r}_i) \equiv m(1, \vec{r}_{cm}) \qquad \vec{p}_{cm} \equiv \sum_i \vec{p}_i = \sum_i \vec{p}_{i\,cm}$$

$$= \underbrace{\sum_i \vec{r}_{cm} \times \vec{p}_{i\,cm}} + \vec{r}_{cm} \times \cancel{\vec{p}_i{}'} + \cancel{\vec{r}_i{}' \times m_i} \dot{\vec{r}}_{cm} + \underbrace{\vec{r}_i{}' \times \vec{p}_i{}'}$$

$$= \vec{L} = \vec{r}_{cm} \times \vec{p}_{cm} \quad + \quad \vec{S} = \sum_i \vec{r}_i{}' \times \vec{p}_i{}'$$

- we will learn QM "addition of angular momenta" later
- occurs even for point-like ("fundamental") particles !
- has the wrong tensorial nature: spin ½ not scalar or vector !
- even spin-1 (which IS a vector) looks different !
- not neccesarily QM - also classical ! (e.g. polarization)
- not a special relativistic effect, even though it appears naturally in the solution of the Dirac eq'n. Levy Leblond, Comm Math Phys 6, 286 (1967)

* QM spin IS:

- intrinsic angular momentum (invariant; nonlocal - independent of $\vec{r}$)
  Ohnian, A.J.P 54 500 (1986), Gsponer, arXiv: physics/0308027 (2003)
- a property of waves ! : it is the "angular momentum / magnetic moment generated by circulation of energy / current in the wave field
  Belinfante, Physica 6 887 (1939), Gordon, Z. Phys 50 630 (1928), Humblet Physica 10, 585 (1943)
- it is caused/characterized by the invariant tensorial nature of the field: a scalar has no spin $S=0$, a vector has $S=1$, Quadrupole/metric tensor $S=2$

eg#1 $\vec{L}$ vector $\rightarrow L_-, L_z, L_+$ $(m=-1,0,1)$ "spherical tensor components"

eg#2 photon: $\vec{E} = (\hat{x} E_x + \hat{y} E_y) e^{ikz - \omega t}$ has 2 polarizations: $\varepsilon_x = \hat{x}$, $\varepsilon_y = \hat{y}$
circular polarization: $\varepsilon_+ = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y})$ and $\varepsilon_- = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y})$ (see Ohnian)
has $S=1$, $M_S = \pm 1$ - note $M_S = 0$ (longitudinal pol) is missing
note: ang. momentum $\frac{\int d\vec{S} \equiv \vec{E} \times \vec{A}\, dV}{\int dU \equiv \frac{1}{2}(\vec{E}\vec{D} + \vec{B}\vec{H})} = \frac{1}{\omega}$ $\quad U = \hbar\omega$ & $S = 1 \cdot \hbar$
$\vec{S} = \vec{r} \times \vec{p} = \int \vec{r} \times \vec{E} \times \vec{B} = \int \vec{E} \times \vec{A}$ quantized for photons.

note: ang. momentum $\frac{\int d\vec{S} \equiv \vec{E}\times\vec{A}\,dV}{\int dU = \frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{B}\cdot\vec{H})} = \frac{1}{\omega}$ $U = \hbar\omega$ & $S = 1\cdot\hbar$

$S = \vec{r}\times\vec{p} = \int \vec{r}\times\vec{E}\times\vec{B} = \int \vec{E}\times\vec{A}$ quantized for photons.

eg#3 $\vec{r}_1 \otimes \vec{r}_2 = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix}\begin{pmatrix} x_2 & y_2 & z_2 \end{pmatrix} = \begin{pmatrix} x_1x_2 & x_1y_2 & x_1z_2 \\ y_1x_2 & y_1y_2 & y_1z_2 \\ z_1x_2 & z_1y_2 & z_1z_2 \end{pmatrix} \sim \underbrace{\vec{r}_1\cdot\vec{r}_2}_{\text{trace scalar } S=0} + \underbrace{\tfrac{1}{2}\vec{r}_1\times\vec{r}_2}_{\text{antisymmetric pseudovector } S=1} + \underbrace{\tfrac{1}{2}(\vec{r}_1\vec{r}_2+\vec{r}_2\vec{r}_1) - \tfrac{1}{3}\vec{r}_1\cdot\vec{r}_2\, I}_{\text{symmetric traceless tensor } S=2}$

- so spin itself is the $\nearrow$ of the fields (multiple components/polarizations) is classical, has geometric interpretation, and can be quantized.

- perhaps the real mystery is that $S = \frac{1}{2}, \frac{3}{2}, \ldots$ exist in nature, as predicted by the operator analysis of angular momentum.

- so the question isn't "what is spin", but rather "what is the geometrical interpretation of a 2-component field?"

convention answer: "an irreducible representation of the rotation group" ie. it just mathematically works out that way. (from operator analysis)

* Each fundamental particle is characterized by its spin:

BOSONS $S=0$: Higgs H, scalar mesons π, η, K $S=1$: vector bosons/mesons $\gamma, W^\pm, Z, g$, $\rho, \omega, \ldots$ $S=2$: graviton $g_{\mu\nu}$

FERMIONS $S=\frac{1}{2}$: quarks & leptons $\begin{matrix} u & c & t \\ d & s & b \end{matrix}$ $\begin{matrix} e & \mu & \tau \\ \nu_e & \nu_\mu & \nu_\tau \end{matrix}$ nucleons (octet) $p, n, \Sigma, \Lambda, \Xi$ $S=3/2$: (decuplet) $\Delta, \Sigma, \Xi, \Omega$

* Spin-½ system: 1) $\vec{S}\times\vec{S} = i\hbar\vec{S}$ [angular momentum]
Postulates 2) $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ is an intrinsic property of each particle

- for $s = \frac{1}{2}$ we can't write a wave function, we don't want to anyway.
- the state is: $|sm\rangle$ defined as the simultaneous eigenstate of:

$$S^2|sm\rangle = \hbar^2 s(s+1)\,|sm\rangle \rightarrow \tfrac{3}{4}\hbar^2 \quad (s=\tfrac{1}{2})$$
$$S_z|sm\rangle = \hbar m\,|sm\rangle \qquad m = \pm\tfrac{1}{2}$$

$$S_\pm|sm\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}\,|s, m\pm1\rangle \rightarrow \begin{cases} 0 & \text{off the edge} \\ \hbar & \text{raising/lowering} \end{cases}$$

$S_\pm |s\, m\rangle = \hbar\sqrt{s(s+1) - m(m\pm 1)}\; |s, m\pm 1\rangle \rightarrow \begin{cases} 0 & \text{off the edge} \\ \hbar & \text{raising/lowering} \end{cases}$

used to get $S_x \pm i S_y = S_\pm$

* cannonical matrix representation of $s = 1/2$: the SPINOR

ie $\vec{V} = V_x \hat{x} + V_y \hat{y}$

$\chi = a\chi_+ + b\chi_-$ $\qquad \chi_+ \equiv |\tfrac{1}{2}\, \tfrac{+1}{2}\rangle \qquad \chi_- \equiv |\tfrac{1}{2}\, \tfrac{-1}{2}\rangle$

$\begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$S^2\begin{pmatrix} a \\ b \end{pmatrix} = 3/4\,\hbar\begin{pmatrix} a \\ b \end{pmatrix}$ $\qquad$ so $S^2 = 3/4\,\hbar\, I$

PAULI MATRICES

$S_z\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{+1}{2}\hbar\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$S_z\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{-1}{2}\hbar\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\qquad$ so $S_z = \frac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\qquad \sigma_z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$S_+\left(\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right) = \hbar\left(\begin{array}{c|c} 0 & 1 \\ 0 & 0 \end{array}\right)$ $\qquad$ so $S_+ = \hbar\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\qquad \sigma_+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$

$S_-\left(\begin{array}{c|c} 1 & 0 \\ 0 & 1 \end{array}\right) = \hbar\left(\begin{array}{c|c} 0 & 1 \\ 1 & 0 \end{array}\right)$ $\qquad$ so $S_- = \hbar\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$

$S_x = \frac{1}{2}(S_+ + S_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\qquad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$S_y = \frac{1}{2i}(S_+ - S_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

check: $S_x^2 + S_y^2 + S_z^2 = S_\pm S_\mp \pm \hbar S_z + S_z^2 = S^2 = 3/4\,\hbar^2\, I$

* eigenvectors of σ_x or σ_y

$\sigma_x \chi_\pm^{(x)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \lambda\begin{pmatrix} a \\ b \end{pmatrix}$ $\qquad \begin{vmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{vmatrix} = \lambda^2 - 1 = 0 \qquad \lambda = \pm 1$

$\lambda = 1$: $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad \lambda = -1$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$U^{(x)} = \left(\chi_+^{(x)} \; \chi_-^{(x)}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad \boxed{\sigma_x U^{(x)} = U^{(x)} \underbrace{\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}}_{\sigma_z}}$$

* Pauli algebra — a "Clifford algebra".

- let $\vec{\sigma} \equiv [\sigma_x, \sigma_y, \sigma_z]$ then $\sigma_i \sigma_j = \underbrace{\delta_{ij} I}_{\text{DOT (sym)}} + \underbrace{i \varepsilon_{ijk} \sigma_k}_{\text{CROSS. (antisym}}$

  "basis frame"! $\qquad (\vec{\sigma}\cdot\vec{A})(\vec{\sigma}\cdot\vec{B}) = \vec{A}\cdot\vec{B}\, I + i\vec{\sigma}\cdot\vec{A}\times\vec{B}$

- the fact that $\vec{\sigma}\times\vec{\sigma} = 2i\vec{\sigma}$ mimics the cross product allows $\vec{\sigma}$ to "generate" rotations.

- decomposition of $GL_2(\mathbb{C})$ 2x2 complex matrices:

$$M = \underbrace{I a}_{\text{Scalar}} + \underbrace{\vec{\sigma}\cdot\vec{b}}_{\text{Vector}} + \underbrace{i\vec{\sigma}\cdot\vec{c}}_{\text{Axial vector}} + \underbrace{i I d}_{\text{Pseudoscalar}}$$

* what is the state of an electron?

$$|n l m_l m_s\rangle = |n l m_l\rangle \otimes |s m_s\rangle \cong \phi_{nlm_l}(r,\theta,\phi) \otimes \chi_{m_s} = \begin{pmatrix} \psi^+_{nlm} \\ \psi^-_{nlm} \end{pmatrix}(r,\theta,\phi)$$

$$|\psi\rangle = \sum_{m_s=\pm 1/2} \sum_{nlm} c^{m_s}_{nlm_l} \phi^{m_s}_{nlm_l}(r,\theta,\phi)\, \chi_{m_s}$$

- It is a "spinor field" — 2-component wave function with certain transformation properties.

- What are these transformation properties?
- What is the geometrical significance of them?