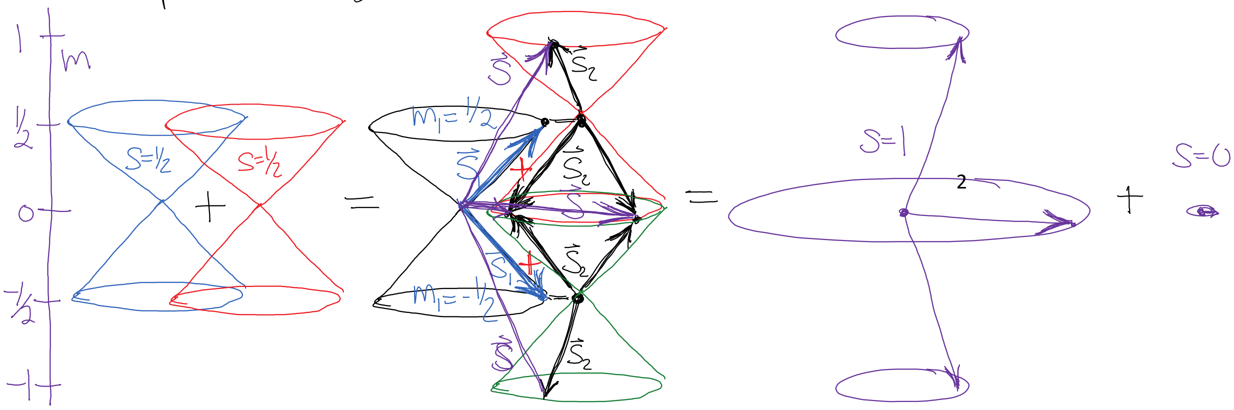
L56-Addition of Angular Momenta

Friday, February 5, 2016 08:17

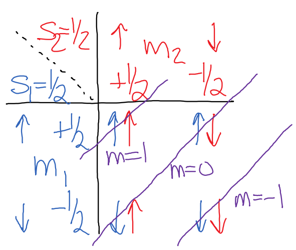
* classically $\vec{L} = \vec{L}_1 + \vec{L}_2$
  just adds component by component

* quantum mechanically, only one component is observable.
  however the operators still add: $\vec{J} = \vec{J}_1 + \vec{J}_2$

  - example: $\vec{S}_1 + \vec{S}_2$ where $s_1 = 1/2$ $s_2 = 1/2$



* what do the states look like?
  it is a "tensor product" of the two spaces:
  - all possible combinations of basis states



"addition" of spin produces
② $\vec{S}_1$ states × ② $\vec{S}_2$ states
= ④ "product" states:

$$|s_1 m_1\rangle \otimes |s_2 m_2\rangle = |m_1 m_2\rangle \in \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$$

note $|\uparrow\downarrow\rangle$ and $|\downarrow\uparrow\rangle$ are different states!

* how do you operate on these states?

  $\vec{S}_1$ acts on $|s_1 m_1\rangle$ and ignores $|s_2 m_2\rangle$ (like a constant in front of the operator)

  $\vec{S}_2$ ignores $|s_1 m_1\rangle$ and acts on $|s_2 m_2\rangle$

$$(S_z = S_{1z} + S_{2z})\underbrace{|s_1 m_1\rangle|s_2 m_2\rangle}_{\text{these do not act on eachother}} = \underbrace{(S_{1z}|s_1 m_1\rangle)}_{\hbar m_1 |s_1 m_1\rangle}|s_2 m_2\rangle + |s_1 m_1\rangle\underbrace{(S_{2z}|s_2 m_2\rangle)}_{\hbar m_2 |s_2 m_2\rangle}$$

$$m = m_1 + m_2 \qquad = \hbar\underbrace{(m_1 + m_2)}_{m}|s_1 m_1\rangle|s_2 m_2\rangle$$

$$S_z|\uparrow\uparrow\rangle = \hbar\underbrace{(\tfrac{1}{2} + \tfrac{1}{2})}_{}|\uparrow\uparrow\rangle \qquad S_z|\uparrow\downarrow\rangle = \hbar\underbrace{(\tfrac{1}{2} - \tfrac{1}{2})}_{}|\uparrow\downarrow\rangle$$

$$\overbrace{\quad\quad}^{m}$$

$$S_z|\uparrow\uparrow\rangle = \hbar\underbrace{\left(\tfrac{1}{2}+\tfrac{1}{2}\right)}_{m=1}|\uparrow\uparrow\rangle \qquad S_z|\uparrow\downarrow\rangle = \hbar\underbrace{\left(\tfrac{1}{2}-\tfrac{1}{2}\right)}_{m=0}|\uparrow\downarrow\rangle$$

$$S_z|\downarrow\uparrow\rangle = \hbar\underbrace{\left(-\tfrac{1}{2}+\tfrac{1}{2}\right)}_{m=0}|\downarrow\uparrow\rangle \qquad S_z|\downarrow\downarrow\rangle = \hbar\underbrace{\left(-\tfrac{1}{2}-\tfrac{1}{2}\right)}_{m=-1}|\downarrow\downarrow\rangle$$

* can we form multiplets out of these states with $m = -s, \dots s-1, s$?

$m = 1, \boxed{0}, -1$ corresponds to $s=1$ "triplet"
$m = \boxed{0}$ corresponds to $s=0$ "singlet"

which state goes where ?

– note that $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$ are both symmetric w/r particle exchange $1\leftrightarrow 2$ (more about this next chapter)

the linear combination $\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right)$ is also symmetric

thus the triplet is $\left\{\uparrow\uparrow, \frac{1}{\sqrt{2}}(\uparrow\downarrow+\downarrow\uparrow), \downarrow\downarrow\right\}$

the singlet is antisymmetric: $\frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$ has $s=0$

* confirmation 1: try ladder operators: $S_\pm = S_{1\pm} + S_{2\pm}$

$$S_-\uparrow\uparrow = (S_-\uparrow)\uparrow + \uparrow(S_-\uparrow) \propto \downarrow\uparrow + \uparrow\downarrow$$

$$S_-\left([\uparrow\downarrow]\pm[\downarrow\uparrow]\right) \propto \left[(S_-\uparrow)\downarrow + \uparrow\cancel{(S_-\downarrow)}\right] \pm \left[\cancel{(S_-\downarrow)}\uparrow + \downarrow(S_-\uparrow)\right] \propto \downarrow\downarrow \pm \downarrow\downarrow \propto \begin{matrix}\downarrow\downarrow\\ 0\end{matrix}$$

* confirmation 2: calculate $S^2 = (\vec{S}_1+\vec{S}_2)^2 = \underbrace{S_1^2}_{\hbar^2\frac{1}{2}(\frac{3}{2})} + 2\vec{S}_1\cdot\vec{S}_2 + \underbrace{S_2^2}_{\hbar^2\frac{1}{2}(\frac{3}{2})}$

$$S_1\cdot S_2 = \frac{\hbar^2}{4}\vec{\sigma}_1\cdot\vec{\sigma}_2 = \frac{\hbar^2}{4}\left(\sigma_{1x}\sigma_{2x} + \sigma_{1y}\sigma_{2y} + \sigma_{1z}\sigma_{2z}\right)$$

$$\sigma_{1x}\sigma_{2x}|\uparrow\uparrow\rangle = (\sigma_{1x}\uparrow)(\sigma_{2x}\uparrow) = \downarrow\cdot\downarrow = |\downarrow\downarrow\rangle$$

$$\sigma_{1y}\sigma_{2y}|\uparrow\uparrow\rangle = (\sigma_{1y}\uparrow)(\sigma_{2y}\uparrow) = i\downarrow\cdot i\downarrow = -|\downarrow\downarrow\rangle$$

these cancel.

$$\sigma_{1z}\sigma_{2z}|\uparrow\uparrow\rangle = (\sigma_{1z}\uparrow)(\sigma_{2z}\uparrow) = \uparrow\cdot\uparrow = |\uparrow\uparrow\rangle$$

thus $S^2|\uparrow\uparrow\rangle = \hbar^2\left(\frac{1}{2}\cdot\frac{3}{2} + 2\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{3}{2}\right)|\uparrow\uparrow\rangle = \hbar^2\underbrace{(1)}_{s=1}(2)|\uparrow\uparrow\rangle$

others: $\vec{\sigma}\cdot\vec{\sigma}\,|\uparrow\downarrow\rangle = \downarrow\uparrow + i(-i)\downarrow\uparrow + 1\cdot(-1)\uparrow\downarrow = 2|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle$

$$\vec{\sigma}\cdot\vec{\sigma}\,|\downarrow\uparrow\rangle = \uparrow\downarrow + (-i)i\uparrow\downarrow + -1\cdot 1\uparrow\downarrow = 2|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$$

$$\vec{\sigma}\cdot\vec{\sigma}\,|\downarrow\downarrow\rangle = \uparrow\uparrow + (-i)(-i)\uparrow\uparrow + -1\cdot-1\downarrow\downarrow = |\downarrow\downarrow\rangle$$

$\vec{\sigma}\cdot\vec{\sigma}\ |\downarrow\uparrow\rangle = \uparrow\downarrow + (-i)(i)\uparrow\downarrow + -1\cdot 1\ \uparrow\downarrow = 2|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$

$\vec{\sigma}\cdot\vec{\sigma}\ |\downarrow\downarrow\rangle = \uparrow\uparrow + (-i)(-i)\uparrow\uparrow + -1\cdot -1\ \downarrow\downarrow = |\downarrow\downarrow\rangle$

thus

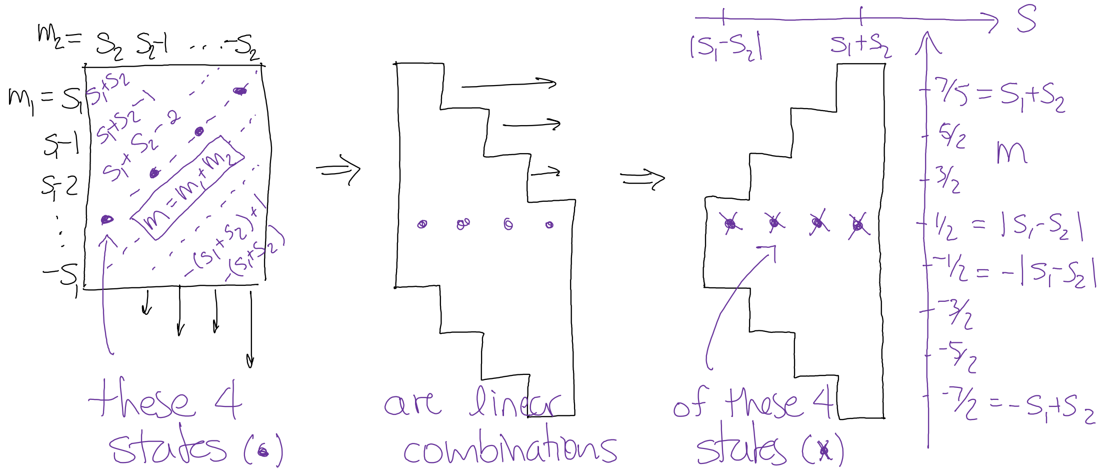
$$S^2|\uparrow\uparrow\rangle = \hbar^2\cdot 2|\uparrow\uparrow\rangle$$
$$S^2|\uparrow\downarrow\rangle = \hbar^2\cdot(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)$$
$$S^2|\downarrow\uparrow\rangle = \hbar^2\cdot(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)$$
$$S^2|\downarrow\downarrow\rangle = \hbar^2\cdot 2|\downarrow\downarrow\rangle$$

$$S = \begin{array}{c|cccc|c} & \uparrow\uparrow & \uparrow\downarrow & \downarrow\uparrow & \downarrow\downarrow & \\ \hline \uparrow\uparrow & 2 & & & & m=1 \\ \uparrow\downarrow & & 1 & 1 & & m=0 \\ \downarrow\uparrow & & 1 & 1 & & \\ \downarrow\downarrow & & & & 2 & m=-1 \\ \hline & m=1 & m=0 & & m=-1 & \end{array}$$

diagonalize centre block:

$$\begin{pmatrix}1&1\\1&1\end{pmatrix}\underbrace{\begin{pmatrix}1\\1\end{pmatrix}}_{\frac{1}{\sqrt{2}}(\uparrow\downarrow+\downarrow\uparrow):\ S=1} = 2\begin{pmatrix}1\\1\end{pmatrix} \qquad \begin{pmatrix}1&1\\1&1\end{pmatrix}\underbrace{\begin{pmatrix}1\\-1\end{pmatrix}}_{\frac{1}{\sqrt{2}}(\uparrow\downarrow-\downarrow\uparrow):\ S=0} = 0\begin{pmatrix}1\\-1\end{pmatrix}$$

* in general: $S_1 + S_2$ for arbitrary S_1, S_2



* the transformation matrix between these independent subspaces are called "Clebsch-Gordon coefficients"

rpp2011-rev-clebsch-gordan-coefs

http://pdg.lbl.gov/2011/reviews/rpp2011-rev-clebsch-gordan-coefs.pdf

36. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND *d* FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for $-8/15$ read $-\sqrt{8/15}$.

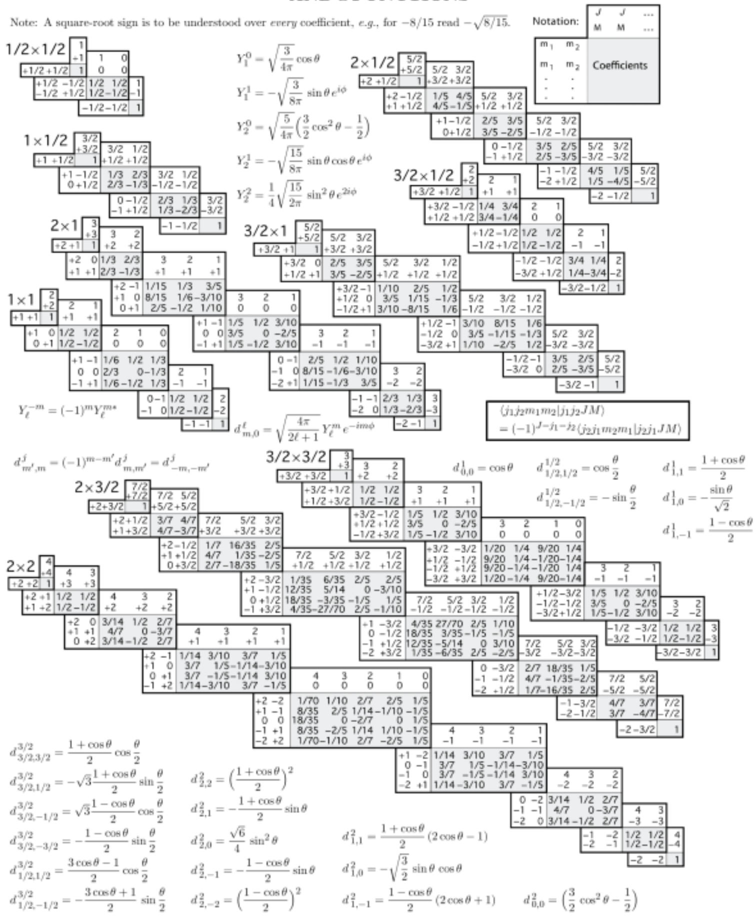


Figure 36.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).