L69-Fine Structure

Friday, March 11, 2016 07:07

* Hydrogen atom $\mathcal{H} = -\frac{\hbar^2}{2m}\nabla^2 - \underbrace{\frac{e^2}{4\pi\epsilon_0}}_{\text{electric}}\frac{1}{r}$ $\quad \underbrace{\hbar c}_{\text{quantum}} \approx 197\,\text{eV}\cdot\text{nm}$

$$a = \frac{4\pi\epsilon_0\hbar^2}{me^2} = \frac{\hbar c}{\alpha \cdot mc^2} \approx \frac{(\text{quantum})^2}{\text{electric}\cdot\text{mass}} \qquad \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137.036} \approx \frac{\text{electric}}{\text{quantum}}$$

$$E_1 = -\frac{m}{2\hbar^2}\cdot\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 = -\frac{1}{2}mc^2\cdot\alpha^2 \approx \text{mass}\cdot\left(\frac{\text{electric}}{\text{quantum}}\right)^2$$

$mc^2 = 0.511$ MeV (rest-mass energy)

$m_p c^2 = 938$ MeV (proton rest mass)

- energy scales
  - Bohr energies: $\alpha^2 mc^2$
  - Fine structure: $\alpha^4 mc^2$ (F.S.) $\sim 10^{-4}$
  - Lamb shift: $\alpha^5 mc^2$ (L.S.) $\sim 10^{-6}$
  - Hyperfine structure: $\frac{m_e}{m_p}\alpha^4 mc^2$ (H.F.S.) $\sim 10^{-7}$

  } perturbations.

- today we will study the first perturbation: fine structure.
  2 contributions: relativistic corrections, spin-orbit coupling.

- note: the exact fine structure can be obtained from the relativistic spin-1/2 Dirac equation!

* relativistic corrections: $T = \frac{1}{2}mv^2 = \frac{p^2}{2m} \rightarrow E^2 - (pc)^2 = (mc^2)^2$

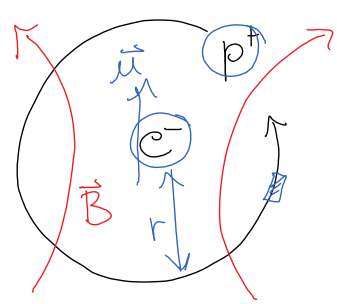
$$T = \sqrt{m^2c^4 + p^2c^2} - mc^2 = mc^2\underbrace{\left(1 + \frac{p^2}{m^2c^2}\right)^{1/2}}_{\text{binomial exp: } (1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{1\cdot 2}x^2 + \dots} - mc^2$$

$$\approx mc^2\left(1 + \frac{1}{2}\frac{p^2}{m^2c^2} + \frac{1/2\cdot -1/2}{1\cdot 2}\left(\frac{p^2}{m^2c^2}\right)^2 + \dots - 1\right) \approx \frac{p^2}{2m} - \frac{1}{8}\frac{p^4}{m^3c^2} + \dots$$

$$\mathcal{H}_r' = \frac{-p^4}{8m^3c^2} \qquad E_r' = \langle \mathcal{H}_r' \rangle = \frac{-1}{8m^3c^2}\langle\psi|p^4|\psi\rangle = \frac{-(E_n)^2}{2mc^2}\left[\frac{4n}{l + 1/2} - 3\right]$$

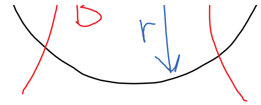
* let's focus on spin-orbit coupling: (you've seen this before!)

$\mathcal{H}' = -\vec{\mu}_e \cdot \vec{B}_p$ $\quad \vec{B}$ field from "orbiting proton"



$$\vec{B}_p = \frac{\mu_0 I}{4\pi}\int\frac{d\vec{l}\times\hat{r}}{r^2} = \frac{\mu_0 I\cdot 2\pi r}{4\pi r^2} = \frac{\mu_0 I}{2r}\hat{L}$$

$I A = \vec{\mu} = \gamma \vec{L} \Rightarrow I = \frac{\gamma}{A} L = \frac{e/2m}{\dots}$



$$I_p A = \vec{\mu}_p = \gamma \vec{L} \quad \Rightarrow \quad I = \frac{\gamma}{A} L = \frac{e/2m}{\pi r^2} L$$

$$\vec{\mu}_e = \gamma \vec{S} = g_e \frac{e}{2m} \cdot \vec{S} \qquad g_e = 2 + \frac{\alpha}{\pi} + \dots \approx 2.002$$

– Thomas precession: the electron is not in an inertial frame, but is accelerating around the proton. This kinematic correction is a factor of 1/2: ($g_e \to g_e - 1$)

thus: $$\mathcal{H}' = \frac{\mu_0}{2r}\left(\frac{e/2m}{\pi r^2}\vec{L}\right)\cdot\left(\frac{e}{m}\vec{S}\right)\cdot\frac{1}{2} = \frac{e^2}{8\pi\varepsilon_0}\cdot\frac{1}{m^2c^2r^3}\vec{S}\cdot\vec{L}$$

$$\left\langle \frac{1}{r^3}\right\rangle = \left[\ell(\ell+1/2)(\ell+1)\, n^3 a^3\right]^{-1} \qquad \langle S\cdot L\rangle = ?$$

* trick for calulating $\langle \vec{S}\cdot\vec{L}\rangle$: $\vec{J} = \vec{L} + \vec{S}$

$$J^2 = L^2 + 2\vec{L}\cdot\vec{S} + S^2 \qquad \vec{L}\cdot\vec{S} = \frac{1}{2}(J^2 - L^2 - S^2) = \frac{\hbar^2}{2}\left(j(j+1) - \ell(\ell+1) - s(s+1)\right)$$

$$E^1_{fs} = E^1_r + E^1_{so} = \frac{(E_n)^2}{2mc^2}\left(3 - \frac{4n}{j+1/2}\right)$$

* we need states with "good" L, S, J quantum numbers!
Review Griffiths 4.4.3 !

in a given orbital "$n\ell$" ie 1s, 2s, 2p, 3s, 3p, 4s, 3d, ...

there are $(g_\ell = 2\ell+1)\cdot(g_s = 2)$ states

$S = 1/2$, $\ell = 2$

m_ℓ \ m_s	$+1/2$	$-1/2$
2	5/2	3/2
1	3/2	1/2
0	1/2	-1/2
-1	-1/2	-3/2
-2	-3/2	-5/2

m_j	$j = 5/2$	$3/2$
5/2	5/2	
3/2	3/2	3/2
1/2	1/2	1/2
-1/2	-1/2	-1/2
-3/2	-3/2	-3/2
-5/2	-5/2	

$\ell = 2$, $s = 1/2$; $m_\ell + m_s = m_j$ always.

The two states $m_\ell = 0$ $m_s = +1/2$ and $m_\ell = 1$ $m_s = -1/2$ are linear combinations of the two states $j = 5/2$ $m_j = 1/2$ and $j = 3/2$ $m_j = 1/2$

Griffiths problem 4.51 Clebsch-Gordon coefficients:

$$|j=l\pm\tfrac{1}{2}, m_j\rangle = \sqrt{\frac{l\pm m_j+1/2}{2l+1}}\,|l, m_l=m_j-1/2\rangle|s=1/2\ m_s=1/2\rangle \pm \sqrt{\frac{l\mp m_j+1/2}{2l+1}}\,|l, m_l=m_j+1/2\rangle|s=1/2, m_s=-1/2\rangle$$