L75-Two Level Systems

Monday, April 4, 2016 07:43

* Time-Dependent Perturbation Theory

- Quantum statics: $\hat{H} = \hat{T} + \hat{V}$ independent of time
  - $\Rightarrow$ TDSE separates into TISE (technique used till now)
  - can solve time dependence of wave function : $e^{-iE_nt/\hbar}$
  - but E_n still constant

- Quantum Dynamics: $\mathcal{H}(t)$
  - needed for transitions (quantum jumps)
  - example: atomic absorpion & emission of radiation

* 2-level system (unperturbed)

let $\mathcal{H} = \mathcal{H}^0 + \mathcal{H}'(t)$ $\quad \mathcal{H}^0 = \hat{T} + \hat{V}^0 \quad \mathcal{H}' =$ time-dependent perturbation potential

$\mathcal{H}^0 \psi_a = E_a \psi_a$ and $\mathcal{H}_0 \psi_b = E_b \psi_b \Rightarrow \langle \psi_i | \psi_j \rangle = \delta_{ij}$

$\psi(0) = c_a \psi_a + c_b \psi_b \Rightarrow \psi(t) = c_a \psi_a e^{-iE_at/\hbar} + c_b \psi_b e^{-iE_bt/\hbar}$

normalization: $\langle \psi | \psi \rangle = |c_a|^2 + |c_b|^2 = 1$

* Perturbed system:

let $\psi(t) = c_a(t) \psi_a e^{-iE_at/\hbar} + c_b(t) \psi_b e^{-iE_bt/\hbar}$ $\quad$ (ψ_a, ψ_b complete basis)

$\mathcal{H}\psi(t) - i\hbar \frac{\partial}{\partial t}\psi = \left[\mathcal{H}^0 - i\hbar \frac{\partial}{\partial t}\right]\psi(t) + \mathcal{H}'(t)\,\psi(t) = 0$

$= \left(-i\hbar \dot{c}_a + c_a \mathcal{H}'\right) \psi_a e^{-iE_at/\hbar} + \left(-i\hbar \dot{c}_b + c_b \mathcal{H}'\right) \psi_b e^{-iE_bt/\hbar} = 0$

since $\left[\mathcal{H}^0 - i\hbar \frac{\partial}{\partial t}\right] \psi_a e^{-iE_at/\hbar} = 0$, likewise for b.

matrix elements: take $\langle \psi_a |$ and $\langle \psi_b |$: $\quad \mathcal{H}'_{ij} = \langle \psi_i | \mathcal{H}' | \psi_j \rangle$

matrix elements: take $\langle \psi_a |$ and $\langle \psi_b |$: $\mathcal{H}'_{ij} = \langle \psi_i | \mathcal{H}' | \psi_j \rangle$

$$\langle \psi_a | -i\hbar \dot{c}_a + c_a \mathcal{H}' | \psi_a \rangle e^{-iE_a t/\hbar} + \langle \psi_a | -i\hbar \cancelto{0}{\dot{c}_b} + c_b \mathcal{H}' | \psi_b \rangle e^{-iE_b t/\hbar} = 0$$

$$i\hbar \dot{c}_a = \cancelto{0}{c_a \mathcal{H}'_{aa}} + c_b \mathcal{H}'_{ab} e^{-i\omega_0 t} \qquad \text{where } \hbar\omega_0 = E_b - E_a$$

typically vanish

$$i\hbar \dot{c}_b = \cancelto{0}{c_b \mathcal{H}'_{bb}} + c_a \mathcal{H}'_{ab} e^{i\omega_0 t} \qquad i\hbar \begin{pmatrix} \dot{c}_a \\ \dot{c}_b e^{-i\omega_0 t} \end{pmatrix} = \begin{pmatrix} \mathcal{H}'_{aa} & \mathcal{H}'_{ab} \\ \mathcal{H}'_{ba} & \mathcal{H}'_{bb} \end{pmatrix} \begin{pmatrix} \dot{c}_a \\ \dot{c}_b e^{-i\omega_0 t} \end{pmatrix}$$

this is an exact equation: TDSE in $\mathcal{H}_0$ basis.

* Perturbation theory: $c_a^{(0)}, c_a^{(1)}, c_a^{(2)}, \ldots$ successive approximations

stick $c_a^{(n)}, c_b^{(n)}$ into RHS, integrate $\int_0^t dt'\, i\hbar \dot{c}_{a,b}$ to get $c_{a,b}^{(n+1)}$ (LHS)

0^{th} order: $c_a^{(0)}(t) = 1 \qquad c_b^{(0)}(t) = 0$ [initial state a]

1^{st} order: $i\hbar \dot{c}_a^{(1)} = \cancelto{0}{c_b^{(0)}} \mathcal{H}'_{ab} e^{-i\omega t} \qquad c_a = 1$

$$i\hbar \dot{c}_b^{(1)} = \cancelto{1}{c_a^{(0)}} \mathcal{H}'_{ba} e^{i\omega_0 t} \qquad c_b^{(1)} = \frac{1}{i\hbar} \int_0^t dt'\, \mathcal{H}'_{ba}(t) e^{i\omega_0 t'}$$

2^{nd} order: $i\hbar \dot{c}_a^{(2)} = c_b^{(1)} \mathcal{H}'_{ab} e^{i\omega_0 t} \qquad c_a^{(2)} = 1 - \frac{1}{\hbar^2} \int_0^t \mathcal{H}'_{ab}(t) e^{-i\omega_0 t'} \left[\int_0^{t'} \mathcal{H}'_{ba}(t'') e^{i\omega_0 t''} dt'' \right] dt'$

* Sinusoidal Perturbations: $\mathcal{H}' = V(\vec{r}) \cos(\omega t)$

$\mathcal{H}'_{ab} = V_{ab} \cos \omega t \qquad$ where $V_{ab} = \langle \psi_a | V | \psi_b \rangle$ as usual

$$c_b^{(1)} = \frac{1}{i\hbar} \int_0^t dt\, V_{ba} \cos \omega t\, e^{i\omega_0 t} = \frac{V_{ba}}{2i\hbar} \int_0^t dt' \left(e^{i(\omega_0+\omega)t'} + e^{i(\omega_0-\omega)t'} \right)$$

$$= \frac{-V_{ba}}{2\hbar} \left[\underbrace{\cancel{\frac{e^{i(\omega_0+\omega)t} - e^0}{\omega_0+\omega}}}_{\text{small}} + \frac{e^{i(\omega_0-\omega)t} - e^0}{\omega_0 - \omega} \right]$$

$$\simeq \frac{V_{ba}}{i\hbar} \frac{\sin(\frac{\omega_0-\omega}{2} t)}{\omega_0 - \omega} e^{i\frac{\omega - \omega_0}{2} t}$$

$$P_{a\to b}(t) = |c_b^{(1)}|^2 = \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2\left(\frac{\omega-\omega_0}{2} t\right)}{(\omega-\omega_0)^2} \quad \text{for} \quad |V_{ab}|^2 \ll \hbar^2(\omega-\omega_0)^2$$

exact solution is called "Rabi flopping" (HW12)