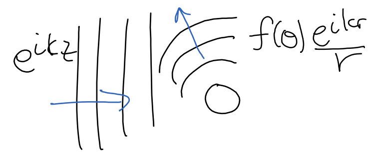
L82-Phase Shifts

Tuesday, April 19, 2016 17:16

* Review and plan for today:

1) Scattering amplitude $f(\theta)$ of cross section $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$

$$\psi(r,\theta) \approx A\left\{ \underbrace{e^{ikz}}_{\text{forward passing}} + \underbrace{f(\theta)}_{\text{scattering amplitude}} \underbrace{\frac{e^{ikr}}{r}}_{\text{outgoing scattering}} \right\}$$



2) Solutions the the Helmholtz (wave) equation:

	HELMHOLTZ (WAVE) OPERATOR	STANDING WAVE		OUTGOING / INCOMING		HARMONIC $k^2=0$ $\nabla^2 f=0$
1-d:	$(\partial_x^2+k^2)$	$[A\cos(kx)+B\sin(kx)]=0$	(OR)	$[Ce^{ikx}+De^{ikx}]$	(OR)	$[A+Bx]$
2-d:	$(\partial_\perp^2+k^2)$	$[A_m J_m(k\rho)+B_m N_m(k\rho)]e^{im\phi}=0$		$[C_m H^{(1)}(k\rho)+D_m H^{(2)}(k\rho)]$		$[A_m\rho^m+B_m\rho^{-m}]$
3-d:	(∂^2+k^2)	$[a_{lm}\, j_l(kr)+b_{lm}\, n_l(kr)]Y_{lm}(\theta\phi)=0$		$[c_{lm} h^{(1)}(kr)+d_{lm} h^{(2)}(kr)]$		$[a_{lm}r^l+b_{lm}r^{-l-1}]$

$$\underbrace{e^{ikz}}_{\text{basis fn}} = \sum_l \underbrace{i^l(2l+1)}_{\text{component}} \underbrace{j_l(kr)\, P_l(\cos\theta)}_{\text{basis fn}}$$

change of basis between plain wave and spherical wave.

3) partial wave amplitudes a_l

using $e^{ikz} = \sum_{l=0}^{\infty} i^l(2l+1)\, j_l(kr)\, P_l(\cos\theta)$ $\qquad (\nabla^2+k^2)j_l(kr)Y_{lm}(\Omega)=0$

and

$$f(\theta)\frac{e^{ikr}}{r} \approx k\sum_{l=0}^{\infty} i^l(2l+1)\, a_l\, h_l^{(1)}(kr)\, P_l(\cos\theta) \qquad f(\theta)=\sum_{l=0}^{\infty}(2l+1)\, a_l\, P_l(\cos\theta)$$

$$\psi_{ext}(\vec r) = A\sum_{l=0}^{\infty} i^l(2l+1)\left[j_l(kr)+ik\, a_l\, h_l^{(1)}(kr)\right]P_l(\cos\theta) \qquad [V(r)=0]$$

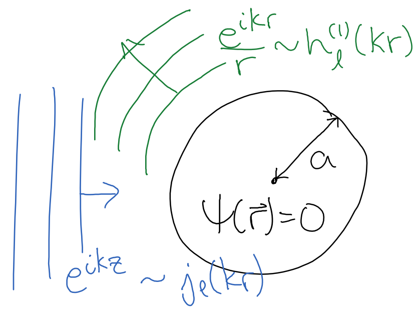
$$\psi_{int}(\vec r) = A\sum_{l=0}^{\infty}\left[b_l\, f_l(r)+c_l\, g_l(r)\right]P_l(\cos\theta)$$ solve a_l using B.C.'s $\qquad [V(r)\neq 0]$

4) TODAY: phase shifts δ_l considering conservation of flux

using $j_l(kr) = \frac{1}{2}\left[\underbrace{h^{(1)}(kr)}_{\text{incoming}} + \underbrace{h^{(1)}(kr)}_{\text{outgoing}}\right] \approx \frac{1}{2k}\left[\underbrace{(-i)^{l+1}\frac{e^{ikr}}{r}}_{\text{incoming}} + \underbrace{i^{l+1}\frac{e^{-ikr}}{r}}_{\text{outgoing}}\right]$

$$\psi_{ext}(\vec r) \approx A\sum_{l=0}^{\infty}\frac{(2l+1)}{2ikr}\left[\underbrace{e^{i(kr+2\delta_l)}}_{j_l(kr)} - \underbrace{(-1)^l e^{-ikr}}_{j_l(kr)}\right]P_l(\cos\theta) \qquad a_l = \frac{e^{i\delta_l}}{k}\sin\delta_l$$

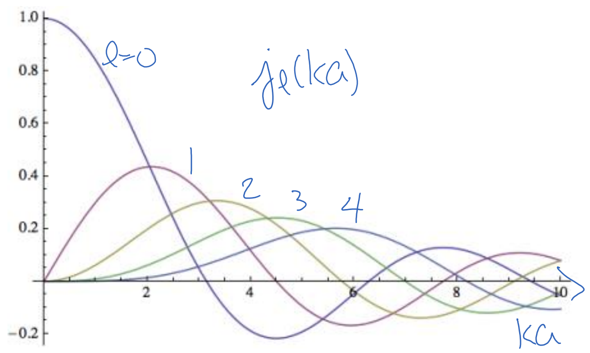
* example 11.3 Hard Sphere Scattering.

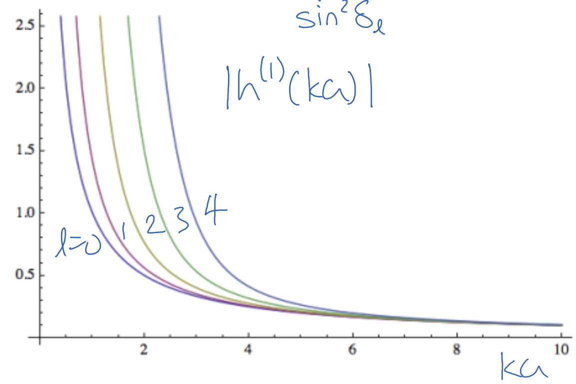


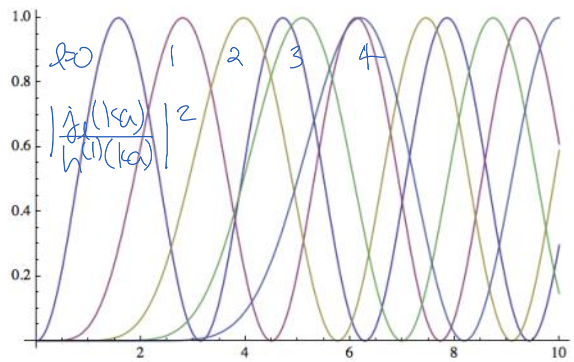
$$\psi(r) = A \sum_{\ell=0}^{\infty} i^{2\ell}(2\ell+1)\left[j_\ell(kr) + ik\, a_\ell h_\ell^{(1)}(kr)\right] P_\ell(\cos\theta)$$

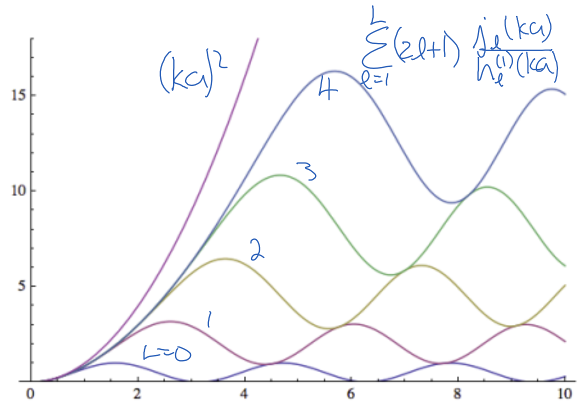
$$\psi(a) = 0 \Rightarrow \left[j_\ell(ka) + ik\, a_\ell h^{(1)}(ka)\right] = 0$$

$$a_\ell = \frac{i\, j_\ell(ka)}{k\, h_\ell^{(1)}(ka)} \qquad \sigma = 4\pi \sum_\ell (2\ell+1)|a_\ell|^2 = \frac{4\pi}{k^2}\sum_\ell (2\ell+1)\underbrace{\left|\frac{j_\ell(ka)}{h_\ell^{(1)}(ka)}\right|^2}_{\sin^2\delta_\ell}$$

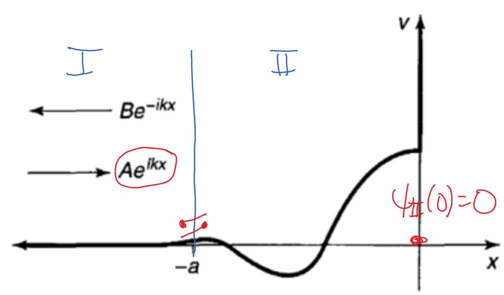








* One-dimensional example:



$\psi_I = Ae^{ikx} + Be^{-ikx}$ $\qquad \psi_{II} = ?$

no transmission: $|A| = |B|$

if $V(x)=0$ then $\psi_I(0) = A + B = 0$ $\qquad B = -A$

$$\psi_0(x) = A\left(e^{ikx} - e^{-ikx}\right) = 2iA\sin(kx)$$

define $B \equiv Ae^{2i\delta}$ so that $\psi(x) \equiv A\left(e^{ikx} - e^{i(2\delta - kx)}\right)$

δ = "phase shift" coming in/going out. $R = |e^{2i\delta}|^2 = 1$
Interference occurs at multiple frequencies.

* Spherical case: each "partial wave" $\psi^{(\ell)}$ reflects independently because a spherically symmetric potential gives $\Delta \ell = 0$.

$$\psi_0 = \Sigma \psi_0^{(\ell)} \qquad \psi_0^{(\ell)} = A i^\ell (2\ell+1)\, j_\ell(kr)\, P_\ell(\cos\theta) \qquad \text{if } V(r) = 0$$

$$j_\ell(x) = \tfrac{1}{2}\left(h^{(1)}(x) + h^{(2)}(x)\right) \approx \frac{1}{2x}\left[(-i)^{\ell+1} e^{ix} + i^{\ell+1} e^{-ix}\right] \qquad x \gg 1$$

$$\psi_0^{(\ell)} = A \frac{2\ell+1}{2ikr}\left(\underbrace{e^{ikr}}_{\text{outgoing.}} - (-1)^\ell \underbrace{e^{-ikr}}_{\text{incoming}}\right) P_\ell(\cos\theta) \quad \text{as } r \to \infty$$

– A potential $V(r)$ $(r<a)$ will shift the outgoing wave by $2\delta_\ell$:

$$\psi^{(\ell)} = A \frac{2\ell+1}{2ikr}\left(e^{i(kr+2\delta_\ell)} - (-1)^\ell e^{-ikr}\right) P_\ell(\cos\theta) \qquad \text{[definition of } \delta_\ell]$$

In terms of incoming plane wave and outgoing spherical wave:

$$\psi = A\left\{e^{ikz} + f(\theta)\frac{e^{ikr}}{r}\right\} = \sum_{\ell=0}^{\infty} \underbrace{A i^\ell (2\ell+1)\left[j_\ell(kr)\right.}_{\psi_0^{(\ell)}} + \underbrace{\left. ik a_\ell h_\ell^{(1)}(kr)\right]}_{\text{scattered wave}} P_\ell(\cos\theta)$$

$$\psi^{(\ell)} - \psi_0^{(\ell)} = A \frac{2\ell+1}{2ikr} e^{ikr}\left(e^{2i\delta_\ell} - 1\right) = A i^\ell (2\ell+1)\, ik a_\ell \underbrace{h_\ell^{(1)}(kr)}_{\frac{1}{kr}(-i)^{\ell+1} e^{ikr}}$$

$$A \frac{2\ell+1}{2ikr} e^{ikr}\left(e^{2i\delta_\ell} - 1\right) = A \frac{2\ell+1}{r} a_\ell e^{ikr}$$

$$a_\ell = \frac{1}{2ik}\left(e^{2i\delta_\ell} - 1\right) = \frac{e^{i\delta_\ell}}{k}\frac{e^{i\delta_\ell} - e^{-i\delta_\ell}}{2i} = \boxed{\frac{e^{i\delta_\ell}}{k}\sin\delta_\ell}$$

← students derived this in class.

$$f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1) a_\ell P_\ell(\cos\theta) = \frac{1}{k}\sum_{\ell=0}^{\infty}(2\ell+1) e^{i\delta_\ell}\sin\delta_\ell\, P_\ell(\cos\theta)$$

$$\sigma = 4\pi\sum_{\ell=0}^{\infty}(2\ell+1)|a_\ell|^2 = \frac{4\pi}{k^2}\sum_{\ell=0}^{\infty}(2\ell+1)\sin^2\delta_\ell$$