

L83-Born Approximation

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* general goal of attack: apply Green's functions to obtain a particular solution of Schrödinger equation with a "source" term. with a couple of hitches

* reminder: we already did this in E&M to solve Poisson's Eq.:

$$-\nabla^2 V = \rho/\epsilon \Rightarrow V = -\nabla^2 \rho/\epsilon = V_p + V_0$$

V_0 is the general solution of $\nabla^2 V = 0$ (homogenous eq.)

• the coefficients of V_0 are constrained to satisfy boundary conditions

V_p is a particular solution to $-\nabla^2 V = \rho/\epsilon_0$ (one of many)

• we solve for V using Green's functions: the solution $-\nabla^2 G(\vec{r}) = \delta^3(\vec{r})$ with a basis function as the source.

$$4\pi G = \frac{1}{r^2} \xrightarrow{\nabla} \frac{\hat{r}}{r^2} \xrightarrow{\nabla} 4\pi \delta^3(\vec{r}) \quad V \xrightarrow{\nabla} \vec{E} \xrightarrow{\nabla} \rho/\epsilon \quad \vec{r} \equiv \vec{r} - \vec{r}'$$

• the solution is a linear combo of basis solutions indexed by $\vec{r}'$ (for a point charge at $\vec{r}'$): $\rho(\vec{r}) = \int d^3r' \underbrace{\delta^3(\vec{r} - \vec{r}')}_{\text{basis}} \underbrace{\rho(\vec{r}')}_{\text{component}}$

$$\begin{aligned} V_p(\vec{r}) &= -\nabla^2 \rho(\vec{r})/\epsilon = -\nabla_{\vec{r}}^2 \left[\int d^3r' \delta^3(\vec{r} - \vec{r}') \rho(\vec{r}') \right] / \epsilon \\ &= \int d^3r' \rho(\vec{r}') \left[-\nabla_{\vec{r}}^2 \delta^3(\vec{r} - \vec{r}') = G(\vec{r} - \vec{r}') = \frac{1}{4\pi\epsilon_0 r} \right] = \int \frac{d^3r' \rho(\vec{r}')}{4\pi\epsilon_0 r} \end{aligned}$$

Example: Helmholtz theorem: $\nabla^2 = \nabla \cdot \nabla - \nabla \times \nabla \times$

$$\begin{aligned} \vec{F} &= -\nabla \left(\underbrace{-\nabla^2 \vec{F}}_V \right) + \nabla \times \left(\underbrace{-\nabla^2 \vec{F}}_{\vec{A}} \right) \\ -\nabla^2 V &= \rho/\epsilon \quad V = -\nabla^2 \rho/\epsilon = \int \frac{d^3r' \rho(\vec{r}')}{4\pi\epsilon_0 r} \quad -\nabla^2 \vec{A} = \mu \vec{J} \quad \vec{A} = \nabla^2 \mu \vec{J} = \int \frac{\mu d^3r' \vec{J}(\vec{r}')}{4\pi r} \end{aligned}$$

* Application to Schrödinger Equation:

$$-\frac{\hbar^2}{2m} \nabla^2 \psi + V(\vec{r}) \psi = E \psi \quad \xrightarrow{V=0} (\nabla^2 + k^2) \psi = 0 \quad E = \frac{\hbar^2 k^2}{2m}$$

plane waves: homogenous solution

$$(\nabla^2 + k^2) \psi = Q = \frac{2mV}{\hbar^2} \psi$$

wave operator note that "source" term has $\psi(r)$ embedded!

Two "minor" differences: (latter causes major difficulties!)

- a) need Green's function for Helmholtz, not Laplace eq'n:
 $k^2 \approx -m^2$ called "mass term": creates waviness or attenuation
 note: $k^2 = 0$ for E&M: the photon is massless.

$$(\nabla^2 + k^2) G(\vec{r}) = \delta^3(\vec{r}) \quad \text{where} \quad G(r) = -\frac{e^{ikr}}{4\pi r}$$

is solved in Fourier-momentum space: $\nabla e^{i\vec{q} \cdot \vec{r}} = i\vec{q} e^{i\vec{q} \cdot \vec{r}}$

$$\psi(\vec{r}) = \psi_0(\vec{r}) - \frac{m}{2\pi\hbar^2} \int d^3r_0 \frac{e^{ik(\vec{r}-\vec{r}_0)}}{|\vec{r}-\vec{r}_0|} V(\vec{r}_0) \psi(\vec{r}_0)$$

- b) the source term depends on the wave function:
 use perturbation theory: $(\nabla^2 + k^2) \psi^{(n)} = \frac{2mV}{\hbar^2} \psi^{(n-1)}(r)$
 (the Born Series).

$$\psi_0 = e^{ikz} \rightarrow Q_0 \xrightarrow{(\nabla^2 + k^2)^{-1}} \psi_1 \rightarrow Q_1 \xrightarrow{(\nabla^2 + k^2)^{-1}} \psi_2 \dots$$