L84-Born Approximation (PWIA)

Wednesday, April 27, 2016 08:00

* Summary: last class we learned how to solve "source" terms (no dependent variable) in PDE's using Green's functions

$$-\nabla^2 V(\vec{r}) = \rho(\vec{r})/\varepsilon \quad \Rightarrow \quad V_0 = R(r)P(\theta)Q(\phi) \quad , \quad \nabla^2 G(\vec{r}) = \delta^3(\vec{r})$$

$$V(\vec{r}) = \sum_{l,m} [A_{lm} r^l + B_{lm} r^{-l-1}] Y_{lm}(\theta,\phi) + \int \left\{ d^3r' \left[G(\vec{r}) = \frac{1}{4\pi r} \right] \frac{\rho(\vec{r}')}{\varepsilon} = \frac{dq'}{4\pi\varepsilon r} \right\}$$

* Now we turn Schrödinger's PDE into an integral equation using the Green's function of the Helmholtz (wave) operator

- The scattering potential can be treated perturbatively as a source if it is low energy: $V_0 \ll \frac{\hbar^2 k^2}{2m}$

$$\underbrace{(\nabla^2 + k^2)}_{\text{wave operator}} \psi(\vec{r}) = \frac{2mV}{\hbar^2}\psi(\vec{r}) \equiv \underbrace{Q(\vec{r})}_{\text{source}} \quad [\text{TISE}] \quad \text{where } E = \frac{\hbar^2 k^2}{2m}$$

$$\psi(\vec{r}) = (\nabla^2 + k^2)^{-1} Q(\vec{r}) = \psi_0(\vec{r}) + \int d^3r_0\, G(\vec{r}-\vec{r}_0) \cdot Q(\vec{r}_0)$$

$$= \psi_0(\vec{r}) + \int d^3\vec{r}_0 \frac{e^{ik|\vec{r}-\vec{r}_0|}}{4\pi|\vec{r}-\vec{r}_0|} \frac{2m}{\hbar^2} V(\vec{r}_0)\psi(\vec{r}_0) \quad , \quad \text{where}$$

1) $\psi_0(\vec{r}) = \sum_{l,m} A_{lm}\, j_{lm}(k\vec{r})\, Y_{lm}(\theta,\phi) \to Ae^{ikz}$ is the solution of $(\nabla^2 + k^2)\psi_0(\vec{r}) = 0$, the homogeneous (wave) equation, the arbitrary "constant of 'integration'" used to satisfy B.C.'s.

2) $G(r) = \frac{ik}{4\pi} h_0^{(1)}(kr) + \psi_0(\vec{r})$ are Green's functions, solutions of $(\nabla_{\vec{r}}^2 + k^2)\, G(\vec{r}-\vec{r}_0) = \delta^3(\vec{r}-\vec{r}_0)$ for arbitrary $(\nabla^2+k^2)\psi_0 = 0$.

ie. $h_0^1(x) = \frac{-ie^{ix}}{x}$ is just one of many possibilities. It is a "particular"ly useful one: an outgoing wave $\left(\frac{e^{ikr}}{r}\right)$

Note: $G(\vec{r})$ is the "bad" solution of $(\nabla^2+k^2)\psi(\vec{r})=0$,

with a singularity as $\vec{r}\to 0$ to give the $\delta(\vec{r})$ source term.

* Today we will apply two approximations:

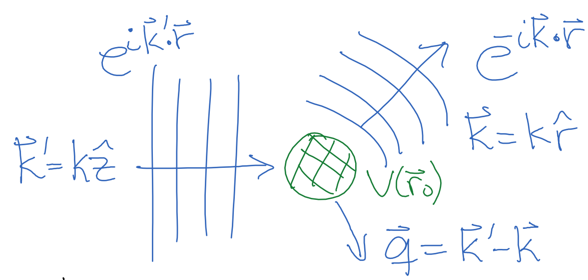
1) the "impulse approximation" $G(\vec{r}) \approx \frac{-e^{ikr}}{4\pi r} e^{i\vec{k}\cdot\vec{r}}$ to recover the outgoing wave and $f(\theta)$ for the cross section. (exact in the limit $r \gg r_0$)

2) the 1st Born "plane wave approximation" $\psi(\vec{r}_0) \approx \psi_0 = A e^{ikz}$, the 1st term in the perturbative expansion, for a definite incoming wave (valid if $V(\vec{r}) \ll \frac{\hbar^2 k^2}{2m}$)

These two, collectively known as PWIA (plane wave impulse approx). give simple incoming and outgoing states, allowing us to interpret scattering as a time-dependent perturbation with the transition matrix element $\langle k | V | k' \rangle$. Other approximations: Distorted (DWIA) or Coulomb (CWIA).

It is also very easy to include higher order terms, which we will do with Feynman diagrams and use them to motivate Quantum Field Theory (QFT) or, the "2nd quantization", where Green's functions will take on the role of "propagators" of the force (quantum field)

* Scattering amplitude: $\vec{r} \to \infty$



for a confined potential $V(\vec{r}_0)$,

$$|\vec{r}-\vec{r}_0|^2 \approx r^2 - 2\vec{r}\cdot\vec{r}_0 \approx r^2\left(1 - 2\frac{\vec{r}\cdot\vec{r}_0}{r^2}\right) \quad \text{so} \quad |\vec{r}-\vec{r}_0| \approx r - \hat{r}\cdot\vec{r}_0$$

let $k = k\hat{r}$ then $G = \frac{-e^{ik|\vec{r}-\vec{r}_0|}}{4\pi|\vec{r}-\vec{r}_0|} \approx \frac{-e^{ikr}}{4\pi r} e^{-i\vec{k}\cdot\vec{r}_0}$

+ Boundary conditions: $\psi_i = A e^{ikz}$ [incoming flux]

+ Boundary conditions: $\psi_0 = Ae^{ikz}$ [incoming flux]

+ then $$\psi(\vec{r}) \cong Ae^{ikz} + \underbrace{\left[\frac{-m}{2\pi\hbar^2}\int d^3\vec{r}_0\, e^{-i\vec{k}\cdot\vec{r}_0}\, V(\vec{r}_0)\, \psi(\vec{r}_0)\right]}_{f(\theta)} \frac{e^{ikr}}{r}$$

$f(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\langle e^{-i\vec{k}\cdot\vec{r}} | V(\vec{r}) | \psi(\vec{r})\rangle$ where $\hat{k}$ points in (θ,ϕ) direction

This is the transition amplitude from $\psi(r)$ to $e^{i\vec{k}\cdot\vec{r}}$

* Born approximation: $\psi(\vec{r}_0) \approx \psi_0(\vec{r}_0) = Ae^{ikz} = Ae^{i\vec{k}'\cdot\vec{r}}$

where $\vec{k}' = k\hat{z}$ [Note: other texts reverse $\vec{k} \leftrightarrow \vec{k}'$!]

then $$f(\theta,\phi) \approx \frac{-m}{2\pi\hbar^2}\langle e^{-i\vec{k}\cdot\vec{r}} | V(r_0) | e^{i\vec{k}'\cdot\vec{r}}\rangle = \frac{-m}{2\pi\hbar^2}\int d^3\vec{r}_0\, e^{i\vec{q}\cdot\vec{r}_0}\, V(\vec{r}_0)$$

This is the Fourier transform of the potential!

$\hbar(\vec{q} \equiv \vec{k} - \vec{k}')$ is called the momentum transfer.

+ at low energy, $e^{i\vec{q}\cdot\vec{r}_0} \approx 1$ so $f(\theta,\phi) \approx \frac{-m}{2\pi\hbar^2}\int V(\vec{r})\, d^3r$

+ for a symmetric potential, $f(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\int_0^\infty 4\pi r^2 dr\, V(r)\, j_0(kr)$
"Fourier Bessel transform"

* Example 11.4 Optical potential (Finite square well):

$$V(r) = \begin{cases} V_0 & r<a \\ 0 & r>a \end{cases} \qquad f(\theta,\phi) = \frac{-m}{2\pi\hbar^2} V_0 \tfrac{4}{3}\pi a^3$$

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2 = \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2 \qquad \sigma = 4\pi\frac{d\sigma}{d\Omega} = \left(\frac{2mV_0a^3}{3\hbar^2}\right)^2$$

* Formally, $(\nabla^2 + k^2)\psi = V\psi$ with $\frac{\hbar^2}{2m}$ absorbed into V.

$\psi = \psi_0 + g V\psi$, where $(\nabla^2+k^2)\psi_0 = 0$, $(\nabla^2+k^2)g = I$

$$\Psi = \Psi_0 + g V \Psi \ , \quad \text{where} \quad (\nabla^2 + k^2)\Psi_0 = 0 \ , \quad (\nabla^2 + k^2) g = I$$

i.e. $g = \int d^3\vec{r}_0 \, G(\vec{r} - \vec{r}_0) \qquad g f(\vec{r}) = \int d^3\vec{r}_0 \, G(\vec{r} - \vec{r}_0) \, f(\vec{r}_0)$

$$\Psi = (1 - gV)^{-1}\Psi_0 = \Psi_0 + V\Psi_0 + (gV)^2\Psi_0 + (gV)^3\Psi_0 + \dots$$

can also be obtained by itterating $\quad \Psi_n = \Psi_0 + gV\Psi_{n-1}$

$\Psi_1 = \Psi_0 + gV\Psi_0 \qquad \Psi_2 = \Psi_0 + gV\Psi_1 = \Psi_0 + gV\Psi_0 + gVgV\Psi_0$, etc...

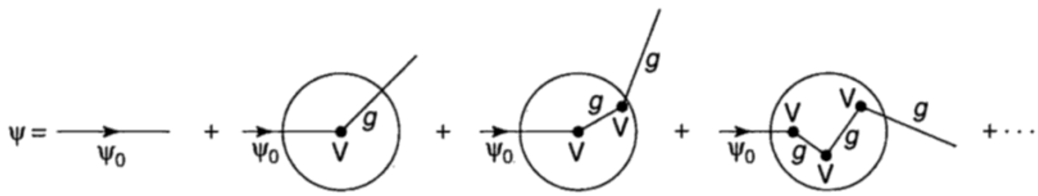


FIGURE 11.13: Diagrammatic interpretation of the Born series (Equation 11.101).