Percolative Conduction and Critical Exponents

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1 Introduction

We are modeling a conductive system as a percolation network to study critical phenomena. In our model of percolation, a network in a square lattice consists of conducting "bonds" or "links" which are joined by "sites" or "nodes". The links between each two sites may be connected with probability p, or not connected with probability 1 – p, and they are assumed to be independent of each other. The possibility that a connected path exists from top to bottom or from side to side is dependent on whether p is above or below the *percolation threshold*, the probability at which a connected path may manifest. Percolation has a singularity at the critical point $p = p_c$ and many properties behave as of a power-law as p approaches p_c. For the square lattice the percolation threshold is exactly $p_c = \frac{1}{2}$. For $p_c < \frac{1}{2}$ there is no such path, and for $p_c > \frac{1}{2}$ there is a greater than zero chance that there is a connected path.

In our conductive system the links represent conductors, with specified conductance; equal to 1(connected link) or 0(missing link). The boundaries(edges) of the square will either be grounded($V = 0$) or not allow current to exit. At a chosen node inside the boundaries the voltage is specified to be $V = 1$. The voltages on the other sites then give rise to currents on the conductive links, and these currents have to balance at every interior site, so that just as much flows into the site as flows out(according to Kirchhoff's Laws). This defines a large set of linear equations that determines the voltages at the interior sites and the total current that flows through the network. From that point we use the set of linear equations to find the total conductance – the total current entering or leaving the network.

Now there are different questions we can pose: one can apply a voltage at one point and provide a grounded point somewhere else, and determine the current; or the grounded part could be one edge of a large square; the top edge could be at one potential and the bottom edge at another; or we could study the case with a rectangle. In each case there has to be long connected paths for there to be conductance; the magnitude of the conductance will depend on the length and whether there are many alternate routes or just a few. We decided to pursue two cases: a lattice with grounded conductors for edges, and a lattice with an insulated edge and the other three edges still being grounded conductors.

2 Square Lattice with all Conducting Edges

2.1 Preliminary Data

The program I will be using to study this model, written by Dr. Straley, has a variable p that is the fraction of internal bonds that have unit conductance. As earlier mentioned, the percolation threshold for the square lattice is exactly $p_c = \frac{1}{2}$, so we will run our simulations with $p = \frac{1}{2}$. This will be our method of studying the percolation problem and study how physical properties behave with system size N. Our first goal is to gather data in the case where we apply a voltage to middle of the lattice and calculate the average conductance with varying sizes of N. From there we will consider other cases that were mentioned above. We plan to run many simulations and gather data using the Lipscomb Compute Cluster(LCC). With our knowledge of the percolation threshold it is possible for us make predictions for how the conductance will behave. The conductivity exponent, t, describes how the electrical conductivity G goes to zero as the percolation threshold is approached when $p > p_c$,

$$G \sim |p - p_c|^t$$

[3]

   With this critical exponent we have the correlation length, ξ, that gives a statistical correlation between two points, contingent on the distance between them.

$$\xi \sim |p - p_c|^{-\nu}$$

[3]

   The exponent, ν, describes the divergence of the correlation length as the percolation threshold is approached. The correlation length tells you the average cluster size, or the average size of one of the irrelevant clusters that isn't connected to the site you are interested in. We can rewrite the conductance laws in term of the distance

$$G \sim \xi^{\frac{-t}{\nu}}$$

From the data we will plot $\log(N)$ versus $\log(G)$ where N is the side length of our square and G is our average conductance. The slope of the curve is the critical exponent, $\frac{-t}{\nu}$, that describes the power law of the conductance with respect to the size of the lattice. The behavior is related to what was described above by replacing ξ by N. Thus, the conductance is given by $G \sim N^{\frac{-t}{\nu}}$. We set the critical exponent equal to $\frac{t}{\nu} \approx 0.972$ [1], although slightly different values have been calculated: we perhaps think it could be $\frac{t}{\nu} = 1$.
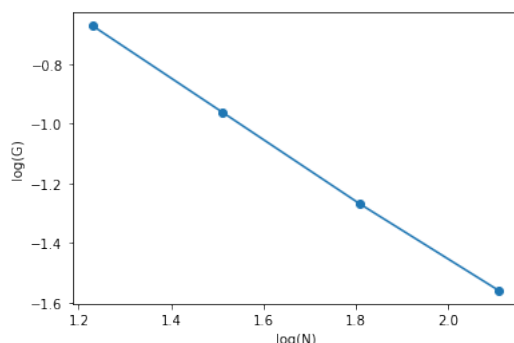
We ran these simulations with N = 16, 32, 65, 128. The slope of the curve is approximately -1.01, which is above our predicted exponent, but memory overflow issues in the program kept us from simulating larger values of N.

2.2 Conductance of Different Size Lattices

To see the relationship between the conductance of the system and the size of the system, where we place the potential in the origin of the lattice, we decided to run the program multiple times with two difference sequences of increasing N. One sequence being even, N = 8, 16, 32, 64, 128, and the other odd, N = 9, 17, 33, 65, 129. The reasoning for one sequence being odd and one sequence being even is because in an even lattice there is no "center". If we have a 4x4 lattice, the site at which we decide the place the potential would slightly be closer to one boundary over the other, whereas in an odd lattice there is a definite origin of the lattice. Although the effects of the off-centered voltage point diminish quickly as the lattice size increases.
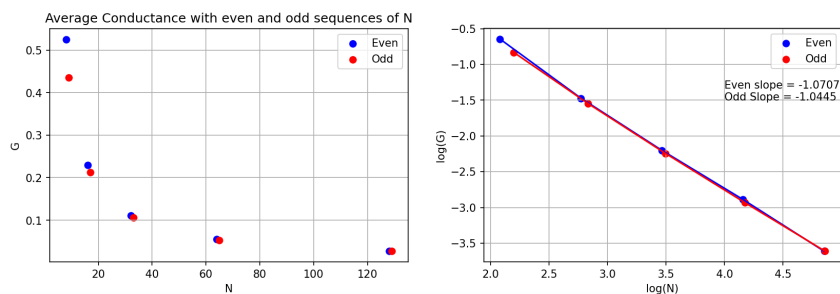
From the plots above we can see quite a difference in the average conductance for lattice sizes 8 and 9, but in the lattice sizes 128 and 129 the difference is not observable(on the graph). We can see that we have slight differences in our exponent, $\frac{t}{\nu}$, in the even and odd runs. The odd run has $\frac{t}{\nu}$ closer to unity than the even run does. This is presumably from the effects the even sequence endures due to the off-centered origin. Now I wanted to observe the behavior of

the conductance with larger lattices knowing that the factors that affect smaller lattices diminish and do not have a significant effect on the conductance in larger cases. In this run I simulated lattices of size N = 50, 100, 150, 200 while still placing the potential in the origin of the lattice.
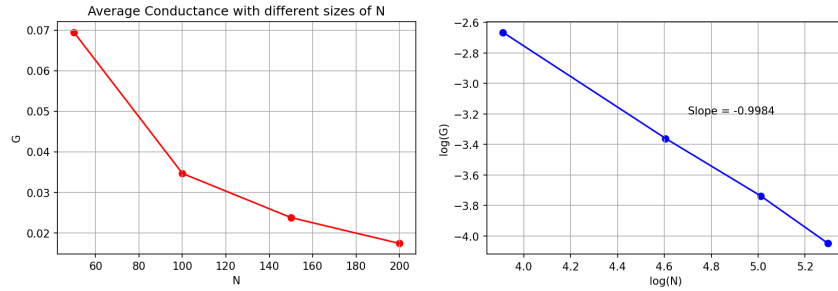

In the above figure our exponent is very close to unity, but our program is limited to only calculated lattices of size 200; therefore we are unable to see how the behavior continues. Possibly in the future the program could be modified to accommodate larger system sizes. With these large lattices we assume the value of the slope would give a more accurate value of $\frac{-t}{\nu}$. In the following weeks I will calculate the lattice that contains a larger set of N sizes, and I hope to increase the size of the lattice beyond the value of 200.

2.3 The Distance Function

In attempting to find the relationship between the average conductance and the distance to the boundary, we want the "distance" that explains the length between a point in the square and the boundary, that treats all sides the same and makes sense from the point of view of conformal transformations. To find this distance function, we want to map our square into the complex plane where the Jacobian elliptic functions produce a repeating lattice of poles and zeros; which we can view as positive and negative charges. We manipulate the elliptic functions to create a square lattice and then map that to an infinite half-plane where we use the method of images technique, which is rather simple for this geometry, and solve for an expression of the distance. Dr. Reš had found an expression for this function [2] to solve a similar problem that Dr. Straley has adjusted to fit our specific problem

$$Distance = \left[\frac{Im[sn([\frac{2K}{N}]z|k)]}{|cn([\frac{2K}{N}]z|k) \times dn([\frac{2K}{N}]z|k)|} \right]$$

Where $z = (x + i(y + K))$ is a complex variable defined on the complex lattice, and sn, cn, and dn are the Jacobian elliptic functions. To further my understanding of this distance concept, I wrote a program in python to plot a contour showing the distance to the boundary at any point in the lattice using the distance function.
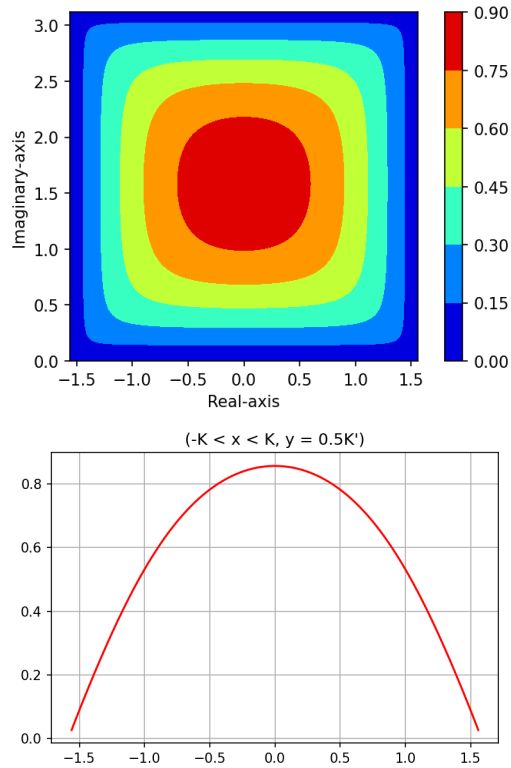
I also made a line graph to show how the distance behaves if you were traverse the lattice from one side to the other.

## 2.4 Conductance at Different Distances from Boundary

After attaining a basic understanding of how the distance formula works, it is time to test it's relation with the conductance. The next part of the project is to calculate the conductance from points at various coordinates in the square, with the square always the same size. If the theory is right, the conductance is a function of the distance only. The two-dimensional set of values, the coordinates of a point in the lattice, collapses into a function of one variable.

For my first run to test this I simulated a lattice of N = 100 at seventeen intentional points which creates a small square lattice of its own.

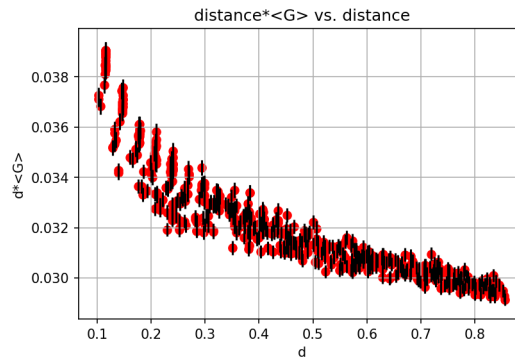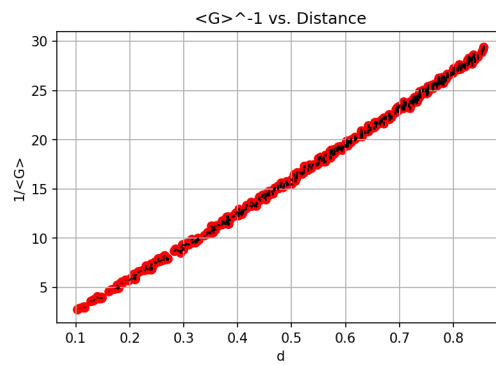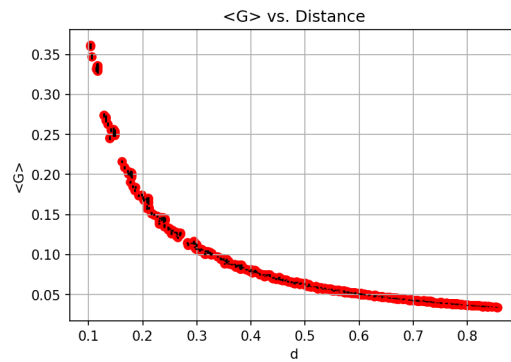Average Conductance vs. Distance
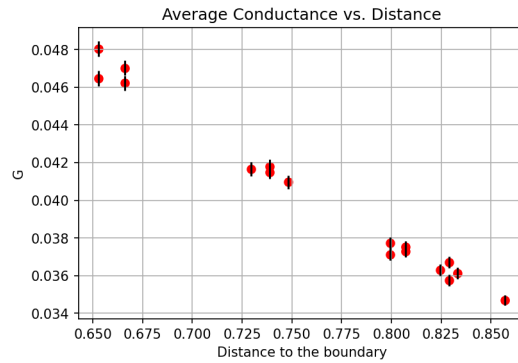
From the plot we can see that the conductance seems to depend on the distance. Also we see multiple points have the different conductance's for the same distance. The reason for this is that in a even numbered lattice there is no proper center as mentioned earlier. So our coordinates are not actually symmetric about the origin.

In my next run I wanted more data to show express the relationship more accurately. I simulated a larger lattice of N = 150 at 40 random coordinates.
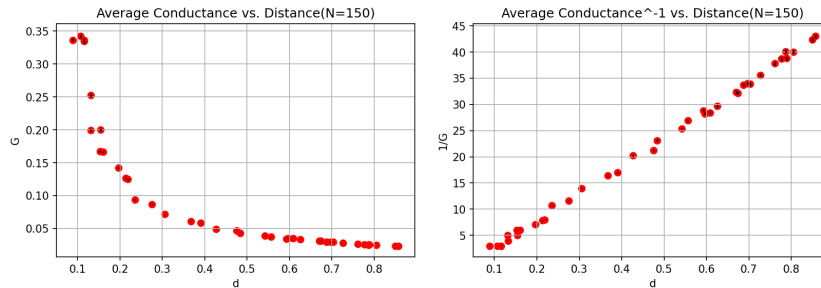
Immediately we get a better picture of a relationship between the conductance and the distance and is clear from these plots, but I decided to increase the sample size by a factor of ten to 500 random points to put the theory to the test.

<G> vs. Distance

<G>^-1 vs. Distance
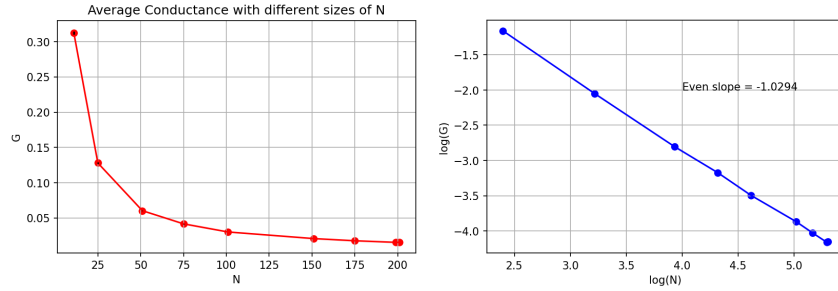

distance*<G> vs. distance

In this plot we can see the conductance relies on the distance more clearly. We can conclude that the conductance to the boundary depends only on this "distance".

3 Square Lattice with one Insulated Edge

3.1 Conductance of Different Size Lattices

In our new configuration, the left edge is insulated ergo no current can leave that edge. To see the relationship between the conductance of the system and the size of the system, where we place the potential in the origin of the lattice, we decided to run the program multiple times with an odd sequence of increasing N = 11,25,51,75,101,151,175,199,201.

Average Conductance with different sizes of N

Our exponent is similar relative to the lattice with all conducting edges which is not too surprising because we expected similar behavior when the voltage point is placed at the origin: where it is the same distance from insulated boundary as the other conducting boundaries. Therefore because we saw this behavior in the previous case, we can reason that we can find a distance function that the conductance is dependent upon.

3.2 The Distance Function

For our new problem we need a new distance function because our "distance" will be different. The insulating edge constraint complicates the geometry of the conformal mapping and consequently the method of images method. In this configuration, the lattice will not preserve the symmetry of its "distance" from the last configuration. Meaning at every point in the lattice there is a different value for this distance. Dr. Straley made a few attempts at creating distance function, but none were successful. The most promising attempt had the form:

$$Distance = \left[\left| \frac{Im(sn(\frac{2K}{N}z|k)) \times (sn(\frac{2K}{N}z|k)}{(cn(\frac{2K}{N}z|k) \times dn(\frac{2K}{N}z|k) \times Re(sn(\frac{2K}{N}z|k))} \right| \right]$$

Again $z = (x + i(y + K))$ is a complex variable defined on the rectangle, and sn, cn, and dn are the Jacobian elliptic functions. This distance function results in the following contour plot, which produces a feasible outcome. The distance becomes infinite for nonzero y as x approaches 0. Though this may be right: when you are close to an insulated edge, all paths into it are blocked, and it might be very unlikely to find a path from close to the insulated edge to a

conducting boundary; the conductance is going to zero, like the inverse of the distance.
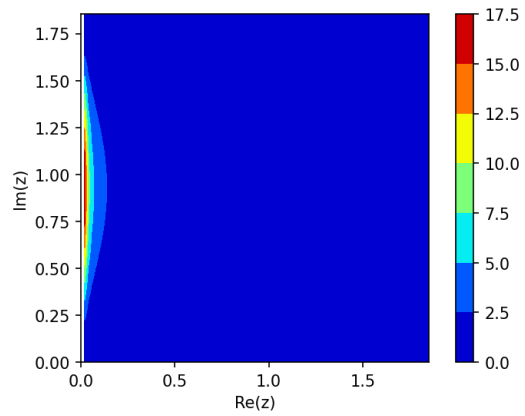
Dr. Straley modified the current program to simulate the new configuration with the left insulated boundary. As before I generated 500 random points in a N = 101 size lattice and found the average conductance at each of those points to test our new function.

Average Conductance vs. Distance

<G>^-1 vs. Distance

Distance*<G> vs. Distance
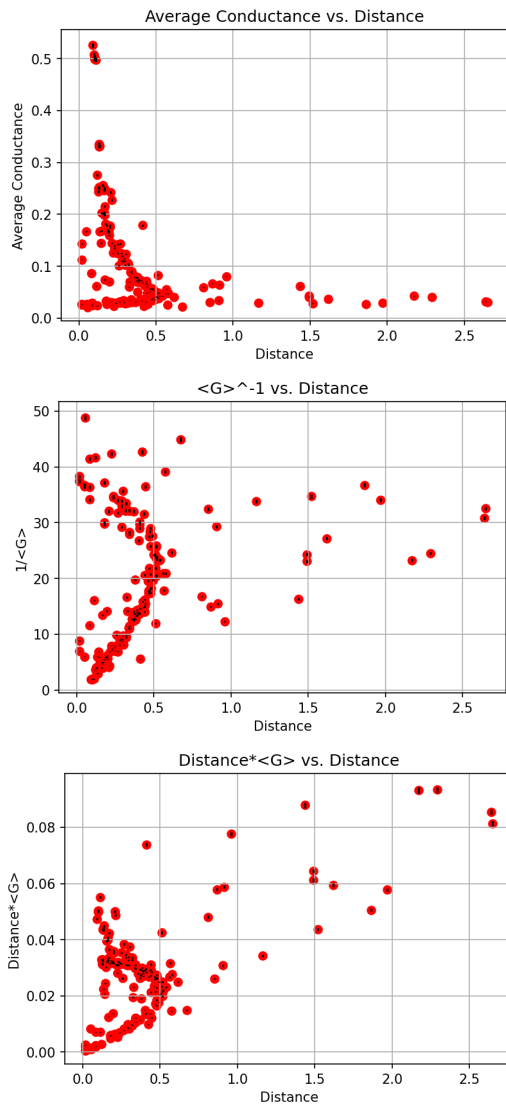
We can somewhat see a relationship, but evident from the plots the relationship between the conductance and the distance is not clearly evident. The function may be correct for coordinates in some vicinity in the lattice but is largely incorrect for the lattice. Further attempts will be made to find the correct function, but we must conclude this distance function is incorrect.

4 Conclusion

In the case of the square lattice with all sides conducting, we have seen that the relationship between the average conductance and the size of the lattice by a power law hold true, and we have seen the same for the conductance and the distance to the boundary. We successfully found a distance function that predicts the value of conduction in a square lattice that is a percolative conducting network at the percolation threshold. In the case of the square with one boundary insulated and the others conductive, the distance to the boundary question can still be studied further, and whether the distance function can be found or not using conformal mapping. Other geometries such as the rectangle or triangle could be explored as well.

5 Acknowledgements

References

[1]   CJ Lobb and DJ Frank. "Percolative conduction and the Alexander-Orbach conjecture in two dimensions". In: *Physical Review B* 30.7 (1984), p. 4090.

[2]   Ivica Reš and Joseph P. Straley. "Order parameter for two-dimensional critical systems with boundaries". In: *Phys. Rev. B* 61 (21 June 2000), pp. 14425–14433. DOI: 10.1103/PhysRevB.61.14425. URL: https://link.aps.org/doi/10.1103/PhysRevB.61.14425.

[3]   Joseph P. Straley. "Percolation Structures and Processes". In: *Annals of the Israel Physical Society* (1983).