Phy 520: Problem Set 3
(Due: October 1, 2018)

1). In this problem we revisit the two-slit experiment with a light beam of energy E (and of "one photon at a time" intensity), using an apparatus that may allow us to determine the slit through which a photon has passed. Our goal is to determine what happens to the interference pattern in this case. Please refer to Fig. 1 (from Cohen-Tannoudji, Diu, Laloë, QM, v.1., p. 50.) in what follows. The center-to-center distance between the slits is a. Note that the plate in which the slits are made is mounted so that it can move in the vertical direction.

a) If $d \gg a$, what is the vertical momentum p_1 transferred to the plate after passage through slit 1 and detection at location x_M (at point M) on the far screen? What is the vertical momentum p_2 transferred to the plate after passage through slit 2 and detection at the same location?

b) If we were to measure the vertical motion of the recoiling screen very precisely, so that the condition $\Delta p \ll |p_1 - p_2|$ holds, what happens to our ability to detect the location x_M? To address this, let Δx be the uncertainty in the determination of x_M. What inequality must Δx satisfy? It is worth noting that the separation of the fringes in the two-slit interference pattern is given by $\lambda d/a$, where $\lambda = hc/E$. How does Δx compare to this quantity? What do you think happens to the interference pattern in this case as a result?

2). Use the uncertainty principle to **estimate** — to an order of magnitude — the maximum time that a sharp, but otherwise ordinary, lead pencil can be balanced upright on its tip. How does the maximum time depend on the sharpness of the pencil? Explain.

(*Hint:* Show that the momentum-position uncertainty relation can be rewritten as $\Delta L \Delta\theta \gtrsim \hbar$, where ΔL is the uncertainty in the angular momentum of the pencil and $\Delta\theta$ is the uncertainty in the angular position of the center of the head of the pencil. For the rough estimate we effect, estimate, it suffices to model the pencil as a mass m on a light rod of length ℓ.)

3). a) Consider Ehrenfest's theorem

$$m\frac{d\langle x\rangle}{dt} = \langle p\rangle\,, \frac{d\langle p\rangle}{dt} = -\langle\frac{dV(x)}{dx}\rangle\,, \tag{1}$$

We demonstrated Eq. (1) in class; prove Eq. (2).

b) A free particle of mass m, moving in one dimension, is described by the following wave packet at time $t = 0$:

$$\psi(x,0) = A\exp(-b|x| + ip_0 x/\hbar)\,. \tag{2}$$

Determine A such that $\psi(x,0)$ is normalized. What are $\langle x\rangle$ and $\langle p\rangle$ at time $t = 0$? How do $\langle x\rangle$ and $\langle p\rangle$ evolve with time? (*Hint:* Note part a)!)

4). a) Consider the wave function

$$\psi(x) = (\alpha/\pi)^{1/4} \exp(-\alpha x^2/2) \tag{3}$$

Compute $\langle x \rangle$ and $\langle x^2 \rangle$. Compute the momentum phase wave function $\psi(p)$, as well as $\langle p \rangle$ and $\langle p^2 \rangle$ using $\psi(p)$. Defining $\Delta x \equiv (\langle x^2 \rangle - \langle x \rangle^2)^{1/2}$ and $\Delta p \equiv (\langle p^2 \rangle - \langle p \rangle^2)^{1/2}$, show that $\Delta x \Delta p = \hbar/2$. As a result, a Gaussian wave function is often called a "minimum uncertainty" wave packet. Generally, we have $\Delta x \Delta p \geq \hbar/2$ — this is the precise statement of the Uncertainty Principle we had promised in lecture.

b) Show that $\Delta x \Delta p > \hbar/2$ for the wave function given in Problem 3b).

5). a) Which of the following operators are linear operators (i.e., in executing $O_i\psi(x)$)? a) $O_1 = x^3$, b) $O_2\psi(x) = x(d/dx)\psi(x)$, c) $O_3 = \lambda K$ (where $K\psi(x) = \psi^*(x)$), d) $O_4\psi(x) = \exp(\psi(x))$, e) $O_5\psi(x) = d\psi(x)/dx + a$, and f) $O_6\psi(x) = \int_{-\infty}^{x} dx'(\psi(x')x')$. Note that λ and a are constants.

b) Compute the commutators $[O_2, O_6]$ and $[O_1, O_2]$. Recall that in so doing we write $[A, B]\psi = A(B\psi) - B(A\psi)$ so that it takes the form $C\psi$.

6). Consider an electron in an infinite box of unknown width. Photons of varying frequencies are observed to be emitted from neighboring energy levels. The largest observed wavelength of these photons is 450 nm. Use this information to determine a, the width of the box.