

Phy 520: Problem Set 6  
(Due: November 14, 2018)

1). Prove the virial theorem in one dimension. Namely, show that

$$\left\langle \frac{p^2}{2m} \right\rangle = \frac{1}{2} \left\langle x \frac{dV}{dx} \right\rangle \quad (1)$$

where  $m$  and  $p$  are the mass and momentum of the particle and  $V(x)$  is the potential. To do this, first show a) that for a real wave function  $\psi(x)$  that

$$\int_{-\infty}^{\infty} dx \psi(x) x \frac{dV(x)}{dx} \psi(x) = -\langle V \rangle - 2 \int_{-\infty}^{\infty} dx \frac{d\psi}{dx} x V(x) \psi(x) \quad (2)$$

and b) use the energy eigenvalue equation to prove that

$$-2 \int_{-\infty}^{\infty} dx \frac{d\psi(x)}{dx} x V(x) \psi(x) = E + \frac{\hbar^2}{2m} \int_{-\infty}^{\infty} dx \left( \frac{d\psi}{dx} \right)^2 \quad (3)$$

Combine these results to prove the virial theorem. How are the expectation values of the potential and kinetic energy operators related if  $V(x) = \alpha x^2$ ?

2). Construct the explicit form of the lowest three eigenfunctions of the harmonic oscillator.

3). Show that if the operator  $A$  is hermitian, then  $\langle A^2 \rangle$  is a positive number.

4). Prove that if  $H$  is a hermitian operator, then the operator  $e^{iH}$  is the hermitian conjugate to  $e^{-iH}$ . *Hint:* Remember that an operator of form  $e^{\pm iH}$  must be interpreted in terms of its power series.

5). An operator  $U$  is said to be unitary if  $UU^\dagger = U^\dagger U = 1$ . Prove that if  $H$  is hermitian, then the operator  $e^{iH}$  is unitary.

6). Prove the Schwartz inequality

$$(\langle \phi | \phi \rangle) (\langle \psi | \psi \rangle) \geq |\langle \phi | \psi \rangle|^2. \quad (4)$$

*Hint:* It is worth noting that  $\langle \chi | \chi \rangle \geq 0$  for any wave function  $\chi$ . Use this relation with  $|\chi\rangle = |\phi\rangle + \lambda|\psi\rangle$ , where  $\lambda$  is a constant, and find its minimum value.

7). Prove the following.

a) If  $A$  and  $B$  are hermitian, then  $i[A, B]$  is also hermitian.

b)  $[AB, C] = A[B, C] + [A, C]B$

c)  $[A, [B, C]] = [B, [C, A]] + [C, [A, B]] = 0$  (This result is known as the Jacobi identity.)

8). Consider a hermitian operator  $H$  that has the property

$$H^4 = 1. \quad (5)$$

What are the eigenvalues of the operator  $H$ ? What are the eigenvalues of  $H$  if  $H$  is not restricted to being hermitian?