Phy 520: Problem Set 7
(Due: November 30, 2018)

1). Using the commutation relations between the operators $\hat{x}$ and $\hat{p}$, determine the equations that describe the time evolution of $\langle x \rangle$ and $\langle p \rangle$ for the Hamiltonian given by

$$H = \frac{p^2}{2m} + \frac{1}{2}m(\omega_1^2 x^2 + \omega_2 x + C) \tag{1}$$

2). Show that (for a Hamiltonian $H = T + V$)

$$\frac{d}{dt}\langle xp \rangle = 2\langle T \rangle - \langle x\frac{dV}{dx} \rangle \tag{2}$$

where T is the kinetic energy. Show for a stationary state (that is, a wave function which does not depend on time) that

$$2\langle T \rangle = \langle x\frac{dV}{dx} \rangle \tag{3}$$

Note that we have now found another pathway to the virial theorem and the result $\langle T \rangle = \langle V \rangle$ for the stationary states of the harmonic oscillator potential.

3). Consider a Hamiltonian whose eigenvectors can be described as a set of states each labeled by a discrete eigenvalue n, that is, $\{|n\rangle\}$, where n is a nonnegative integer. The *trace* of an operator is defined as

$$\mathrm{tr}\, A = \sum_n \langle n|\hat{A}|n\rangle \tag{4}$$

where the sum is over the complete set of states. Use the completeness of the projector P_n (introduced in lecture) to prove that $\mathrm{tr}\, AB = \mathrm{tr}\, BA$.

4). Consider the states $\{|n\rangle\}$ associated with the Hamiltonian with the harmonic oscillator potential. Prove that

$$A|n\rangle = \sqrt{n}|n\rangle\,. \tag{5}$$

5). Consider once again the states $\{|n\rangle\}$ associated with the harmonic-oscillator-potential Hamiltonian and compute $\langle n|\hat{x}^2|n\rangle$, $\langle n|\hat{p}^2|n\rangle$, as well as $(\Delta x)^2$, $(\Delta p)^2$, and $(\Delta x)(\Delta p)$. For what value of n is the uncertainty product minimized?

6). Consider a state vector in the Hilbert space of states associated with the harmonic oscillator Hamiltonian, namely

$$|\psi\rangle = \frac{1}{\sqrt{6}}\begin{pmatrix}1\\2\\1\\0\\.\\.\\.\end{pmatrix} \tag{6}$$

Using that state vector, compute a) $\langle H \rangle$, b) $\langle \hat{x}^2 \rangle$, $\langle \hat{x} \rangle$, $\langle \hat{p}^2 \rangle$, and $\langle \hat{p} \rangle$, and c) $(\Delta x)^2$, $(\Delta p)^2$, and $(\Delta x)(\Delta p)$. Is your result consistent with what you might expect from the Uncertainty Principle? Explain.

Bonus Question (Extra Credit: 10 points): In this problem we derive the form of the minimum-uncertainty wavepacket, recalling the derivation of the uncertainty relation from lecture. To do this, let

$$(\hat{U} + i\lambda_0 \hat{V})|\psi\rangle = 0\,, \tag{7}$$

where $\lambda_0 = -i\langle [\hat{A}, \hat{B}] \rangle / (2(\Delta B)^2)$, $\hat{U} \equiv \hat{A} - \langle \hat{A} \rangle$, $\hat{V} \equiv \hat{B} - \langle \hat{B} \rangle$, where $\hat{A}$ and $\hat{B}$ are Hermitian. Let $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$ and write a differential equation for $\psi(x)$, introducing $\langle \hat{x} \rangle = x_0$ and $\langle \hat{p} \rangle = p_0$. To solve this equation efficiently, let $\psi(x) = e^{ip_0 x/\hbar} f(x - x_0)$ and solve the resulting differential equation in $f(x - x_0)$.