Phy 520: Problem Set 8
(Due: December 7, 2018)

1). *(weighted twice)* Consider a hypothetical quantum mechanical system with just two independent states. Let us describe this system by an orthonormal basis consisting of kets $|1\rangle$ and $|2\rangle$. Suppose $\hat{A}$ is an operator which acts on these states as follows:

$$\hat{A}|1\rangle = \alpha|1\rangle + \beta|2\rangle \tag{1}$$

$$\hat{A}|2\rangle = \gamma|1\rangle \tag{2}$$

where α, β, and γ are complex constants.

a) What is the matrix representation of $\hat{A}$ in this basis? Remember that the $|i\rangle$ are orthonormal. How are the $|i\rangle$ realized in this representation?

b) What is $\langle 1|A^\dagger$? What is $\langle 2|A^\dagger$? (*Hint:* One may readily work out parts b)-d) using the representations determined in part a).)

c) What is $\langle 1|A$? What is $\langle 2|A$?

d) What is $A^\dagger|1\rangle$? What is $A^\dagger|2\rangle$?

e) Find the condition on α, β, and γ such that the operator $\hat{A}$ is hermitian.

f) Find the eigenvalues and eigenvectors of $\hat{A}$. Express the eigenvectors in terms of $|1\rangle$ and $|2\rangle$. Show that if α, β, and γ have been chosen to make $\hat{A}$ hermitian, then the eigenvalues of $\hat{A}$ are real.

g) Find the condition on α, β, and γ such that the operator $\hat{A}$ is unitary.

h) Solve the eigenvalue problem for the operator $\hat{A}^\dagger$. Show that the eigenvalues of $\hat{A}^\dagger$ are the eigenvalues found in part f) if $\hat{A}$ is hermitian. What is the relation between the eigenvalues of $\hat{A}^\dagger$ and the eigenvalues of $\hat{A}$ acting on bras?

2). The angular momentum operator in three spatial dimensions is defined as

$$\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}} \; . \tag{3}$$

a) Prove $[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z$. Use the cyclic permutation of indices to show that this implies $[\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x$ and $[\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y$ as well.

b) Prove $[\hat{\mathbf{L}}^2, \hat{L}_k] = 0$, for $k = x, y, z$.

(*Hint:* Note that $\hat{\mathbf{L}}^2$ does not depend on the coordinate system used to define $\hat{L}_k$, so that it suffices to choose one value of k — your result cannot depend on k.)

3). Consider the matrix representation of the angular momentum operator for a quantum particle in three spatial dimensions.

a) Suppose the particle has an angular momentum of $l = 1$. Use the eigenstates of L_z (denoted as $|1, m_z\rangle$) as a basis and find the matrix representation of the three operators L_x, L_y, and L_z in this three-dimensional subspace. Why is the subspace three-dimensional?

(*Hint*: It is helpful to write L_x, L_y in terms of $L_\pm$.)

b) Suppose the particle has an angular momentum of $l = 1/2$. Use the eigenstates of L_z (denoted as $|1/2, m_z\rangle$) as a basis and find the matrix representation of the three operators L_x, L_y, and L_z in this two-dimensional subspace. Why is the subspace two-dimensional?

(*Hint*: It is helpful to write L_x, L_y in terms of $L_\pm$.)

c) Verify that the matrices you found in part b) satisfy the commutation relation $[L_x, L_y] = i\hbar L_z$.

4). Consider a particle bound in a spherically symmetric potential, for which its state vector has the spatial representation

$$\psi(x, y, z) = C(xy + yz + zx) \exp(-\alpha r^2) \, , \tag{4}$$

where C and α are both real constants. Determine a) the probability that the measurement of $\hat{L}^2$ yields zero, and b) the probability that the measurement of $\hat{L}^2$ yields $6\hbar^2$. c) If the value of ℓ is determined to be 2, then what are the relative probabilities of measuring $m = 2, 1, 0, -1, 2$?

Bonus Question (Extra Credit: 10 points) Imagine a system in which there are *exactly* two linearly independent states:

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |2\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{5}$$

Suppose the Hamiltonian matrix is

$$\mathbf{H} = \begin{pmatrix} h & g \\ g & h \end{pmatrix} . \tag{6}$$

a) Find the eigenvalues and (normalized) eigenvectors of this Hamiltonian.

b) Suppose the system is in state $|1\rangle$ at time $t = 0$. Show that the state of the system at time t is given by

$$|\psi(t)\rangle = \exp(-iht/\hbar) \begin{pmatrix} \cos(gt/\hbar) \\ -i\sin(gt/\hbar) \end{pmatrix} . \tag{7}$$

[The simple two-level system we have discussed provides useful insight into a great variety of physical phenomena. For example, this model can be used to describe **neutrino oscillations**. In this case, we can think of $|1\rangle$ as an electron neutrino (ν_e) and $|2\rangle$ as a muon neutrino (ν_μ); if the system is prepared in a ν_e state at time $t = 0$, the probability with which it can be found in a ν_μ state at time t is non-zero! This phenomenon in matter has been hypothesized as a resolution of the solar neutrino problem; extensive empirical studies have revealed it to be correct.]