Propagation of error for multivariable function

Now consider a multivariable function f(u, v, w,…). If measurements of u, v, w,. All have uncertainty δu, δv, δw, …., how will this affect the uncertainty of the function?

$$\delta f = \left|\frac{\partial f}{\partial u}\right|\delta u + \left|\frac{\partial f}{\partial w}\right|\delta v + \left|\frac{\partial f}{\partial w}\right|\delta w + \cdots \quad \text{(Equation (3.48) of text)}$$

$$\therefore\ f = f(u_0, v_0, w_0, ...\ ) \pm \left(\left|\frac{\partial f}{\partial u}\right|\delta u + \left|\frac{\partial f}{\partial w}\right|\delta v + \left|\frac{\partial f}{\partial w}\right|\delta w + \cdots \right)$$

This works for cases like systematic errors, when the errors of most of the variables have the same sign. For cases like random errors, this overestimate and give an upper bound of the actual error:

$$\delta f \geq \left|\frac{\partial f}{\partial u}\right|\delta u + \left|\frac{\partial f}{\partial w}\right|\delta v + \left|\frac{\partial f}{\partial w}\right|\delta w + \cdots$$

We will study the case of random error later in the course.

Propagation of error for multivariable function
Now the right way: Addition in Quadrature – slide 1

Assuming two random variables, according to the Central Limit Theorem, both x and y will follow Normal distribution. For simplicity, assume their mean equal to 0, i.e. X=Y=0.

$$\Pr ob(x) = \frac{1}{\sigma_X \sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_X{}^2}} \, dx$$

$$\Pr ob(y) = \frac{1}{\sigma_Y \sqrt{2\pi}} e^{-\frac{y^2}{2\sigma_Y{}^2}} \, dy$$

$$\therefore \Pr ob(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left(\frac{y^2}{\sigma_Y{}^2} + \frac{x^2}{\sigma_X{}^2}\right)} \, dxdy$$

Propagation of error for multivariable function
Now the right way: Addition in Quadrature – slide 2

Now consider a new random variable u=x+y and we want to calculate σ_u.
Strategy: Transform x and y into two new variables u =x+y and some other independent variable v=x+py

$$\Pr\mathrm{ob}(u,v) = \frac{1}{\sigma_u\sqrt{2\pi}} e^{-\frac{u^2}{2\sigma_u^2}}\,du$$

$$\Pr\mathrm{ob}(y) = \frac{1}{\sigma_v\sqrt{2\pi}} e^{-\frac{v^2}{2\sigma_v^2}}\,dv$$

$$\therefore \Pr\mathrm{ob}(u,v) = \frac{1}{2\pi\sigma_u\sigma_v} e^{-\frac{1}{2}\left(\frac{u^2}{\sigma_u^2}+\frac{v^2}{\sigma_v^2}\right)}\,du dv$$

$$\text{Compare this with } \Pr\mathrm{ob}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left(\frac{y^2}{\sigma_Y^2}+\frac{x^2}{\sigma_X^2}\right)}\,dx dy$$

$$\therefore \frac{u^2}{\sigma_u^2}+\frac{v^2}{\sigma_v^2} = \frac{y^2}{\sigma_Y^2}+\frac{x^2}{\sigma_X^2}$$

Propagation of error for multivariable function
Now the right way: Addition in Quadrature – slide 3

$$\frac{u^2}{\sigma_u{}^2}+\frac{v^2}{\sigma_v{}^2}=\frac{y^2}{\sigma_Y{}^2}+\frac{x^2}{\sigma_X{}^2} \Rightarrow \frac{(x+y)^2}{\sigma_u{}^2}+\frac{(x+py)^2}{\sigma_v{}^2}=\frac{y^2}{\sigma_Y{}^2}+\frac{x^2}{\sigma_X{}^2}$$

$$\Rightarrow \frac{\sigma_v{}^2(x+y)^2+\sigma_u{}^2(x+py)^2}{\sigma_u{}^2\sigma_v{}^2}=\frac{y^2}{\sigma_Y{}^2}+\frac{x^2}{\sigma_X{}^2}$$

$$\Rightarrow \frac{(\sigma_v{}^2+\sigma_u{}^2)x^2+2(\sigma_v{}^2+p\sigma_u{}^2)xy+(\sigma_v{}^2+p^2\sigma_u{}^2)y^2}{\sigma_u{}^2\sigma_v{}^2}=\frac{y^2}{\sigma_Y{}^2}+\frac{x^2}{\sigma_X{}^2}$$

Compaing coefficient,

$$\frac{\sigma_v{}^2+\sigma_u{}^2}{\sigma_u{}^2\sigma_v{}^2}=\frac{1}{\sigma_x{}^2} \quad \Rightarrow \quad \sigma_v{}^2=\frac{\sigma_x{}^2\sigma_u{}^2}{\sigma_u{}^2-\sigma_x{}^2} \qquad ---(1)$$

$$2(\sigma_v{}^2+p\sigma_u{}^2)=0 \quad \Rightarrow \quad p=-\frac{\sigma_v{}^2}{\sigma_u{}^2} \qquad ---(2)$$

$$\frac{\sigma_v{}^2+p^2\sigma_u{}^2}{\sigma_u{}^2\sigma_v{}^2}=\frac{1}{\sigma_y{}^2} \qquad ---(3)$$

Substitute (2) into (3):

$$\frac{\sigma_v{}^2+\dfrac{\sigma_v{}^4}{\sigma_u{}^2}}{\sigma_u{}^2\sigma_v{}^2}=\frac{1}{\sigma_y{}^2}\Rightarrow\frac{\sigma_u{}^2+\sigma_v{}^2}{\sigma_u{}^4}=\frac{1}{\sigma_y{}^2} \qquad ---(4)$$

Propagation of error for multivariable function
Now the right way: Addition in Quadrature – slide 4

Substitute (1) into (4):

$$\frac{\sigma_u^2 + \dfrac{\sigma_x^2\sigma_u^2}{\sigma_u^2 - \sigma_x^2}}{\sigma_u^4} = \frac{1}{\sigma_y^2} \Rightarrow 1 + \frac{\sigma_x^2}{\sigma_u^2 - \sigma_x^2} = \frac{\sigma_u^2}{\sigma_y^2}$$

$$\Rightarrow \frac{\sigma_u^2}{\sigma_u^2 - \sigma_x^2} = \frac{\sigma_u^2}{\sigma_y^2}$$

$$\Rightarrow \sigma_y^2\sigma_u^2 = \sigma_u^4 - \sigma_x^2\sigma_u^2$$

$$\Rightarrow \sigma_u^2 = \sigma_x^2 + \sigma_y^2$$

Substitute this into (1) $\Rightarrow \sigma_v^2 = \dfrac{\sigma_x^2\sigma_u^2}{\sigma_u^2 - \sigma_x^2} \Rightarrow \sigma_v^2 = \dfrac{\sigma_x^2(\sigma_x^2 + \sigma_y^2)}{(\sigma_x^2 + \sigma_y^2) - \sigma_x^2}$

$$\Rightarrow \sigma_v^2 = \frac{\sigma_x^4 + \sigma_x^2\sigma_y^2}{\sigma_y^2}$$

$$\Rightarrow \sigma_v^2 = \frac{\sigma_x^4}{\sigma_y^2} + \sigma_x^2$$

$$\therefore \quad p = -\frac{\sigma_v^2}{\sigma_u^2} = -\frac{\dfrac{\sigma_x^4}{\sigma_y^2} + \sigma_x^2}{\sigma_x^2 + \sigma_y^2} = -\frac{\dfrac{\sigma_x^4 + \sigma_x^2\sigma_y^2}{\sigma_y^2}}{\sigma_x^2 + \sigma_y^2} = -\frac{\dfrac{(\sigma_x^2 + \sigma_y^2)\sigma_x^2}{\sigma_y^2}}{\sigma_x^2 + \sigma_y^2} = -\frac{\sigma_x^2}{\sigma_y^2}$$

Propagation of error for multivariable function
Now the right way: Addition in Quadrature – slide 5

In summary, we have change from random variables x and y (with uncertainty σ_x and σ_y) to random variables u=x+y and v=x+py (with uncertainty σ_u and σ_v). By simple algebra, we solve the only possibility for this to work:

$$p = -\frac{{\sigma_x}^2}{{\sigma_y}^2}$$

$$\boxed{{\sigma_u}^2 = {\sigma_x}^2 + {\sigma_y}^2}$$

$\leftarrow$ Only this is important because we care only the variable u=x+y but not v.

$${\sigma_v}^2 = \frac{{\sigma_x}^4}{{\sigma_y}^2} + {\sigma_x}^2$$

Therefore, the uncertainty in u=x+y is ${\sigma_u}^2 = {\sigma_x}^2 + {\sigma_y}^2$ (propagation of error).

Rule of adding two errors:

$${\sigma_T}^2 = {\sigma_x}^2 + {\sigma_y}^2 \implies \sigma_T = \sqrt{{\sigma_x}^2 + {\sigma_y}^2}$$

Generalization: Addition in Quadrature

Generalization: Addition of two errors

$$\sigma_T^{\ 2} = \sigma_x^{\ 2} + \sigma_y^{\ 2}$$

Now consider a function of x and y, f(x,y)

$$\text{Error in f due to error in x } (\sigma_x) = \frac{\partial f}{\partial x}\sigma_x$$

$$\text{Error in f due to error in y } (\sigma_y) = \frac{\partial f}{\partial y}\sigma_y$$

$$\boxed{\therefore \text{Error in f} = \sigma_f = \sqrt{\left(\frac{\partial f}{\partial x}\sigma_x\right)^2 + \left(\frac{\partial f}{\partial y}\sigma_y\right)^2}}$$

Propagation of error for multivariable function

Now consider a multivariable function f(u, v, w,…). If measurements of u, v, w,. All have uncertainty δu, δv, δw, …., how will this affect the uncertainty of the function?

$$\delta f = \left|\frac{\partial f}{\partial u}\right|\delta u + \left|\frac{\partial f}{\partial w}\right|\delta v + \left|\frac{\partial f}{\partial w}\right|\delta w + \cdots$$

$$\boxed{\therefore \quad \sigma_f = \sqrt{\left(\frac{\partial f}{\partial u}\sigma_u\right)^2 + \left(\frac{\partial f}{\partial v}\sigma_v\right)^2 + \left(\frac{\partial f}{\partial w}\sigma_w\right)^2 + \cdots}}$$

Formal equation for propagation of error.

Topics

Part 1 – Single measurement

1. Basic stuff (Chapter 1 and 2)
2. Propagation of uncertainties (Chapter 3)

Part 2 – Multiple measurements as independent results

1. Mean and standard deviation (Chapter 4)
2. Basic on probability distribution function (not in text explicitly)
3. The Binomial distribution (Chapter 10)
4. The Poisson distribution (Chapter 11)
5. Normal distribution (first half of Chapter 5)
6. χ^2 test – how well does the data fit the distribution model? (Chapter 12)

Part 3 – Multiple measurements as one sample

1. Central limit theorem (not in text explicitly)
2. Normal distribution (second half of Chapter 5)
3. Propagation of error (Chapter 3)
4. Rejection of data (Chapter 6)
5. Merging two sets of data together (Chapter 7)

Part 4 - Dependent variables

1. Curve fitting (Chapter 8)
2. Covariance and correlation (Chapter 9)

Rejection of data - why

Questions:

Given a set of data $x_1, x_2, ..., x_N$ obtained from N measurements. There is one or two data points lying far apart from the others. We would like to think this is a result of mistake that we did not realize during the measurement. Should we reject this data point from the collection of data?

Example:

Imagine we make six measurement s of the period of a pendulum and get the results (from p. 165 of text):

3.8s, 3.5s, 3.9s, 3.9s, 3.4s, 1.8s.

What are we going to do with the period of 1.8s?

Rejection of data – basic idea

To answer the question quantitatively, we need to investigate the spread of that set of data, this is given by the standard deviation (*not* the standard deviation of mean).

Should we reject the outlying data depends on how far it is away from the mean in comparison with the spread of the data.

Example

Imagine we make six measurement s of the period of a pendulum and get the results (from p. 165 of text):

3.8s, 3.5s, 3.9s, 3.9s, 3.4s, 1.8s.

$$\text{Mean} = \frac{3.8+3.5+3.9+3.9+3.4+1.8}{6} = 3.38$$

Standard deviation

$$= \sigma$$

$$= \sqrt{\frac{(3.8-3.38)^2 + (3.5-3.38)^2 + (3.9-3.38)^2 + (3.9-3.38)^2 + (3.4-3.38)^2 + (1.8-3.38)^2}{5}}$$

$$= 0.8$$

$$\text{SDOM} = \frac{0.8}{\sqrt{6}} = 0.33$$

$$\therefore \mathrm{T} = (3.4 \pm 0.3)\mathrm{s}$$

$\therefore$ 1.8s is 3.38-1.8 =1.58 ~ 2σ from the mean.

Probability of obtaining measurements that differ by at least this much from the mean = 1 – P(less than 2σ)

= 1 –0.95

= 0.05

Chauvenet's criterion for discarding outlying data

Multiply the number of data points N with P. If NP is less than 0.5, the worst data can be discarded. (Chauvenet's criterion).

Example:

Imagine we make six measurement s of the period of a pendulum and get the results (from p. 165 of text):

3.8s, 3.5s, 3.9s, 3.9s, 3.4s, 1.8s.

Probability of obtaining measurements that are worse than the 1.8s data point

= 0.05

$\therefore$ PV = 0.05 $\times$ 6 = 0.3 < 0.5, therefore the 1.8s data point can be rejected.

Example

We can now re-calculate the mean and standard deviation after rejecting the 1.8 s data point:

3.8s, 3.5s, 3.9s, 3.9s, 3.4s, ~~1.8s~~.

$$\text{Mean} = \frac{3.8+3.5+3.9+3.9+3.4}{5} = 3.7$$

Standard deviation

$$= \sigma$$

$$= \sqrt{\frac{(3.8-3.7)^2+(3.5-3.7)^2+(3.9-3.7)^2+(3.9-3.7)^2+(3.4-3.7)^2}{4}}$$

$$= 0.23$$

$$\therefore \ \text{SDOM} = \frac{0.23}{\sqrt{5}} = 0.1$$

$$\therefore \text{T} = (3.7 \pm 0.1)\text{s}$$

A Better Way

Another way is to make more measurements without rejecting any data.

If we do 4 more measurements in the previous example :

3.8s, 3.5s, 3.9s, 3.9s, 3.4s, 1.8s, 3.3s, 3.7s, 4.1s, 3.8s

$$\text{Mean} = \frac{3.8+3.5+3.9+3.9+3.4+1.8+3.3+3.7+4.1+3.8}{10} = 3.52$$

$$\text{Standard deviation}$$

$$= \sigma$$

$$= \sqrt{\frac{(3.8-3.52)^2+(3.5-3.52)^2+(3.9-3.52)^2+(3.9-3.52)^2+(3.4-3.52)^2 +(1.8-3.52)^2+(3.3-3.52)^2+(3.7-3.52)^2+(4.1-3.52)^2+(3.8-3.52)^2}{9}}$$

$$= 0.65$$

$$\therefore \ \text{SDOM} = \frac{0.65}{\sqrt{10}} = 0.2$$

$$\therefore \text{T} = (3.5 \pm 0.2)\text{s}$$