Propagation of error for multivariable function

Now consider a multivariable function f(u, v, w,…). If measurements of u, v, w,. All have uncertainty δu, δv, δw, …., how will this affect the uncertainty of the function?

$$\delta f = \left|\frac{\partial f}{\partial u}\right|\delta u + \left|\frac{\partial f}{\partial w}\right|\delta v + \left|\frac{\partial f}{\partial w}\right|\delta w + \cdots \quad \text{(Equation (3.48) of text)}$$

$$\therefore\ f = f(u_0, v_0, w_0, ...\ ) \pm \left(\left|\frac{\partial f}{\partial u}\right|\delta u + \left|\frac{\partial f}{\partial w}\right|\delta v + \left|\frac{\partial f}{\partial w}\right|\delta w + \cdots \right)$$

This works for cases like systematic errors, when the errors of most of the variables have the same sign. For cases like random errors, this overestimate and give an upper bound of the actual error:

$$\delta f \geq \left|\frac{\partial f}{\partial u}\right|\delta u + \left|\frac{\partial f}{\partial w}\right|\delta v + \left|\frac{\partial f}{\partial w}\right|\delta w + \cdots$$

We will study the case of random error later in the course.

Example

If f = x – y, calculate δf in terms of δx and δy.

$$\delta f = \left|\frac{\partial f}{\partial x}\right|\delta x + \left|\frac{\partial f}{\partial y}\right|\delta y$$

$$= \left|\frac{\partial (x - y)}{\partial x}\right|\delta x + \left|\frac{\partial (x - y)}{\partial y}\right|\delta y$$

$$= |1|\delta x + |-1|\delta y$$

$$= \delta x \ + \ \delta y$$

$$\uparrow$$

Add, *not* subtract

Example

If f = xyz, calculate the fraction error in f.

$$\delta f = \left|\frac{\partial f}{\partial x}\right|\delta x + \left|\frac{\partial f}{\partial y}\right|\delta y + \left|\frac{\partial f}{\partial z}\right|\delta z$$

$$= \left|\frac{\partial (xyz)}{\partial x}\right|\delta x + \left|\frac{\partial (xyz)}{\partial y}\right|\delta y + \left|\frac{\partial (xyz)}{\partial z}\right|\delta z$$

$$= |yz|\delta x + |xz|\delta y + |xy|\delta z$$

$$\therefore \left|\frac{\delta f}{f}\right| = \frac{|yz|\delta x + |xz|\delta y + |xy|\delta z}{|xyz|}$$

$$= \frac{\delta x}{|x|} + \frac{\delta y}{|y|} + \frac{\delta z}{|z|}$$

For product of variables, the resultant fraction error is the sum of the fraction errors of the variables.

Example (Problem 3.7(d) of text)

A student makes the following measurement:

a = 5 ± 1 cm, b = 18 ± 2 cm, c = 12 ± 1 cm, t= 3.0 ± 0.5 s, m= 18 ± 1 gram

Compute the quantity mb/t with its uncertainties and percentage uncertainties

$$f = \frac{mb}{t}$$

$$\delta f = \left|\frac{\partial f}{\partial m}\right|\delta m + \left|\frac{\partial f}{\partial b}\right|\delta b + \left|\frac{\partial f}{\partial t}\right|\delta t$$

$$= \left|\frac{b}{t}\right|\delta m + \left|\frac{m}{t}\right|\delta b + \left|-\frac{mb}{t^2}\right|\delta t$$

$$= \left(\frac{18}{3}\times 1 + \frac{18}{3}\times 2 + \frac{18\times 18}{3^2}\times 0.5\right) \text{cm-g/s}$$

$$= 36 \text{cm-g/s}$$

$$\approx 40 \text{cm-g/s}$$

$$f = \frac{mb}{t} = \frac{18\times 18}{3} = 108 \text{ cm-g/s} \approx 110 \text{ cm-g/s}$$

$$\therefore\ f = \underline{\underline{110 \pm 40 \text{ cm-g/s}}}$$

$$\text{Percentage uncertainty} = \frac{40}{110}\times 100\% = 36\% \approx \underline{\underline{40\%}}$$

Topics

Part 1 – Single measurement

1. Basic stuff (Chapter 1 and 2)
2. Propagation of uncertainties (Chapter 3)

Part 2 – Multiple measurements as independent results

1. Mean and standard deviation (Chapter 4) ←
2. Basic on probability distribution function (not in text explicitly)
3. The Binomial distribution (Chapter 10)
4. The Poisson distribution (Chapter 11)
5. Normal distribution (first half of Chapter 5)
6. χ^2 test – how well does the data fit the distribution model? (Chapter 12)

Part 3 – Multiple measurements as one sample

1. Central limit theorem (not in text explicitly)
2. Normal distribution (second half of Chapter 5)
3. Propagation of error (Chapter 3)
4. Rejection of data (Chapter 6)
5. Merging two sets of data together (Chapter 7)

Part 4 - Dependent variables

1. Curve fitting (Chapter 8)
2. Covariance and correlation (Chapter 9)

Mean and standard deviation

The mean and variance of a collection of data of sample size N are defined as:

$$\bar{x} = \text{mean of sample distribution} = \frac{\sum_{i=1}^{N} x_i}{N}$$

$$\sigma^2 = \text{variance of the measurement}$$

$$= \frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N}$$

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N}} \text{ is called the standard deviation}$$

Example

Consider a collection of data {4.0, 4.5, 3.9, 4.2, 4.3, 3.7}

$$\text{Mean} = \frac{4.0 + 4.5 + 3.9 + 4.2 + 4.3 + 3.7}{6}$$

$$= \frac{24.6}{6} = \underline{\underline{4.1}}$$

$$\text{Standard deviation} = \sqrt{\frac{(4.0 - 4.1)^2 + (4.5 - 4.1)^2 + (3.9 - 4.1)^2 + (4.2 - 4.1)^2 + (4.3 - 4.1)^2 + (3.7 - 4.1)^2}{6}}$$

$$= \sqrt{\frac{0.42}{6}} = \sqrt{0.07}$$

$$= \underline{\underline{0.26}}$$

$\therefore$ We expect the next measurement will probably fall in the range of

4.1 ± 0.26

Two Definitions of Standard Deviation

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N}}$$ is called the *population* standard deviation

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N - 1}}$$ is called the *sample* standard deviation

For large N, population standard deviation ≈ sample standard deviation. At this point, we will use population standard deviation, but later sample standard deviation will become more important.

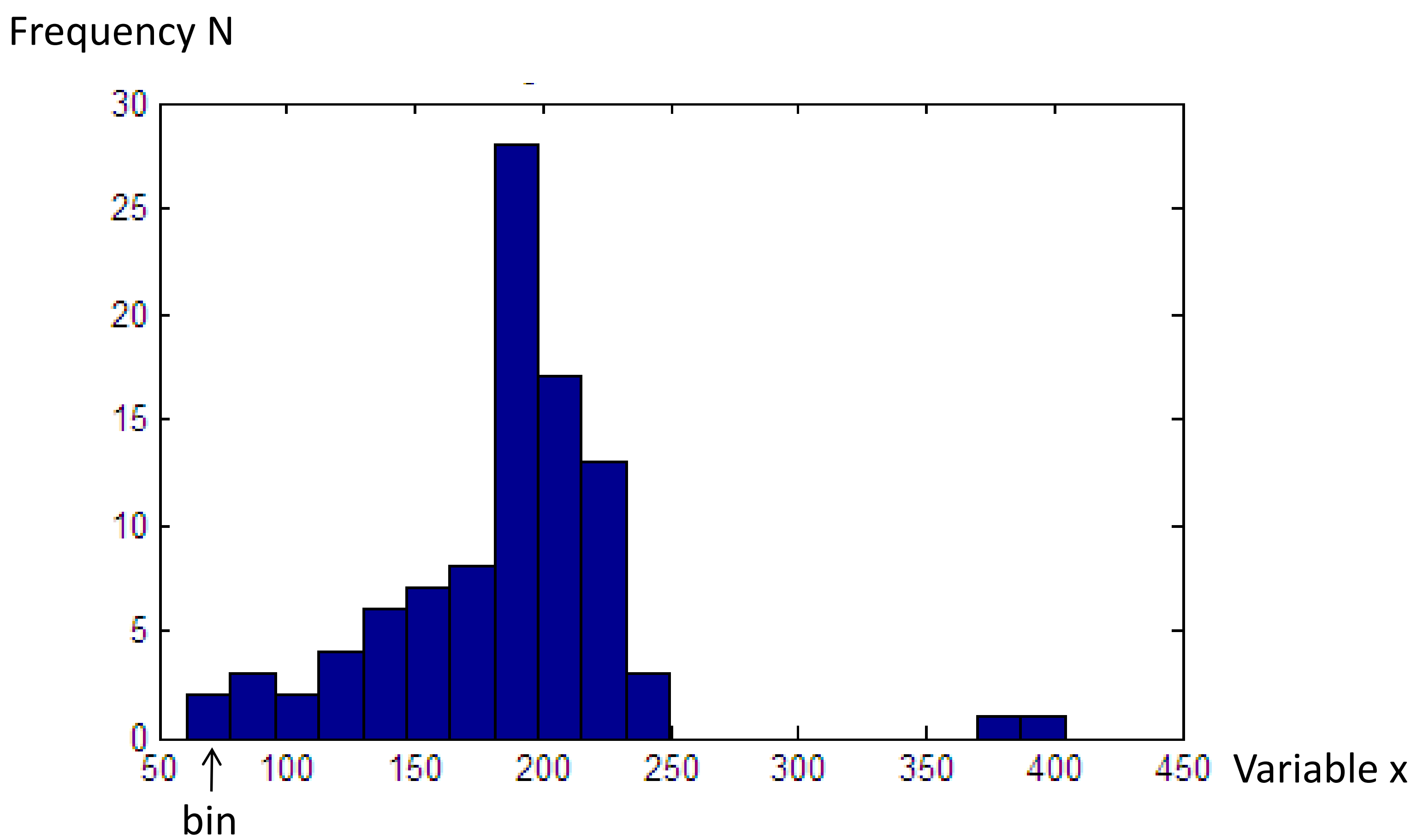
Population Standard Deviation

Population standard deviation $\sigma = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N}}$ is a measure

of the actual *spread* of the data.

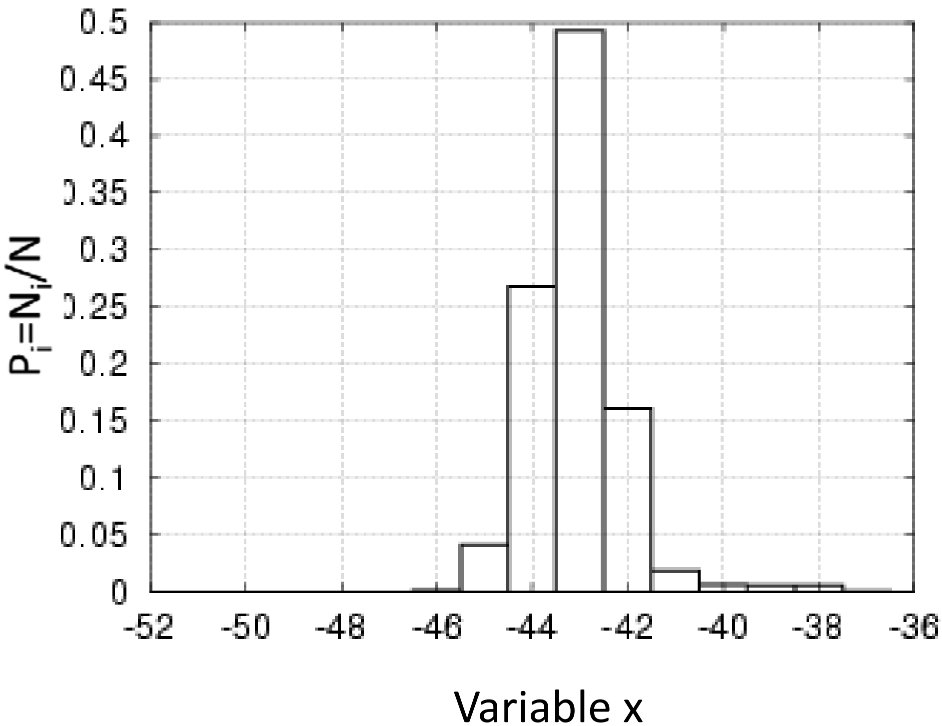
When the sample size is N=1, the population standard deviation equals to 0, because there is no spread in the data. However, this does not mean this is a good collection of data.

Histogram



Assuming all bins have the same size, if you do one more measurement, probability for its value to fall in bin i = $P_i = \frac{N_i}{\sum_j N_j} = \frac{N_i}{N}$. This is probably the best you can guess from the data.

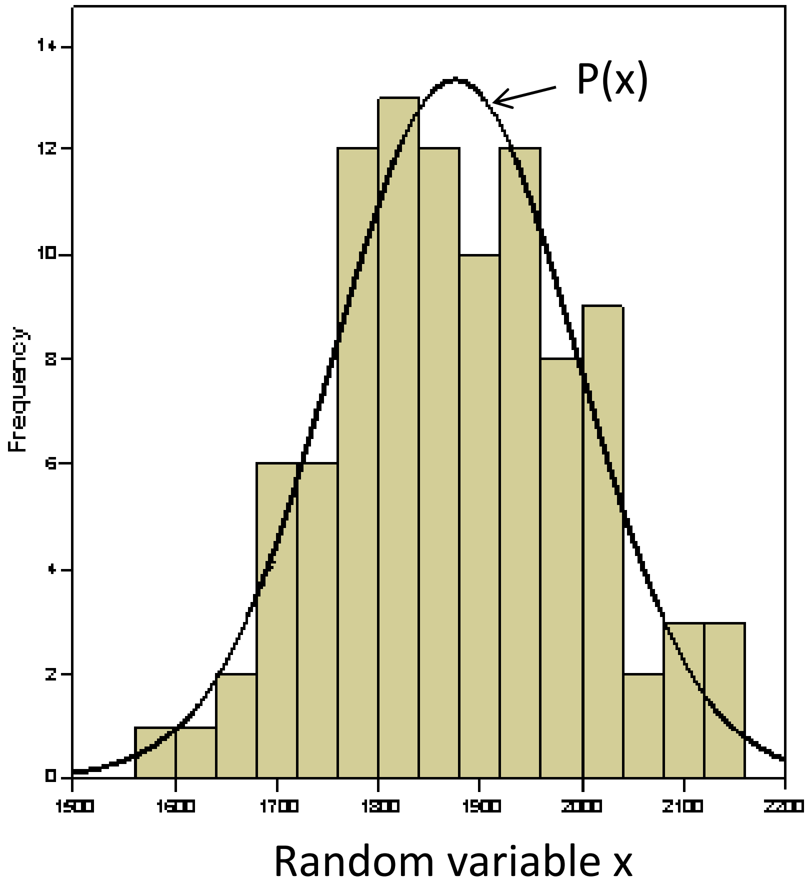
Normalized Histogram



Assuming all bins have the same size, you can just read Pi from the vertical axis of a *normalized* histogram.

Note that $\sum_j P_j = 1$

Probability Density Function



When N $\rightarrow\infty$, we can make the bin size $\rightarrow$ 0 (=dx). When this happens, the normalized histogram will become a *probability density function*:

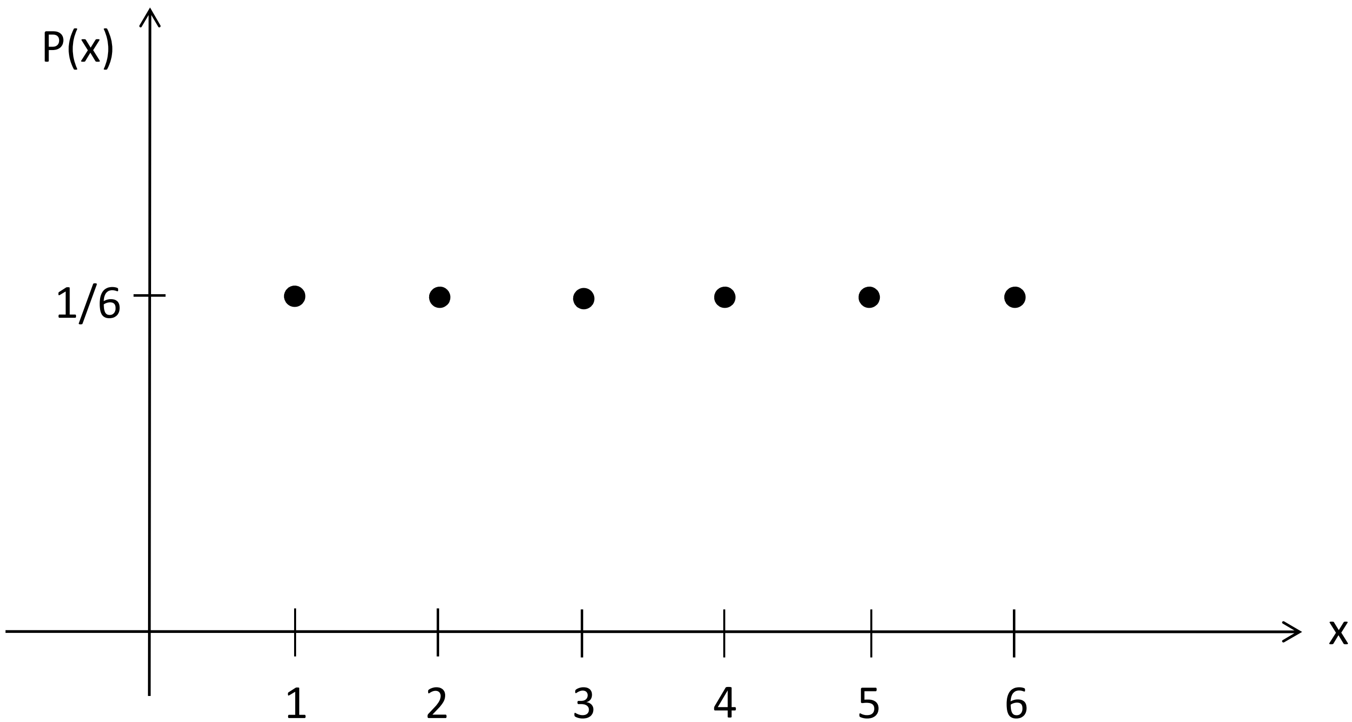
$P_i \rightarrow P(x)dx$

$$\sum_j P_j = 1 \Rightarrow \int P(x)dx = 1$$

Unfortunately, we will never know P(x) because it is impossible to satisfy N $\rightarrow\infty$.

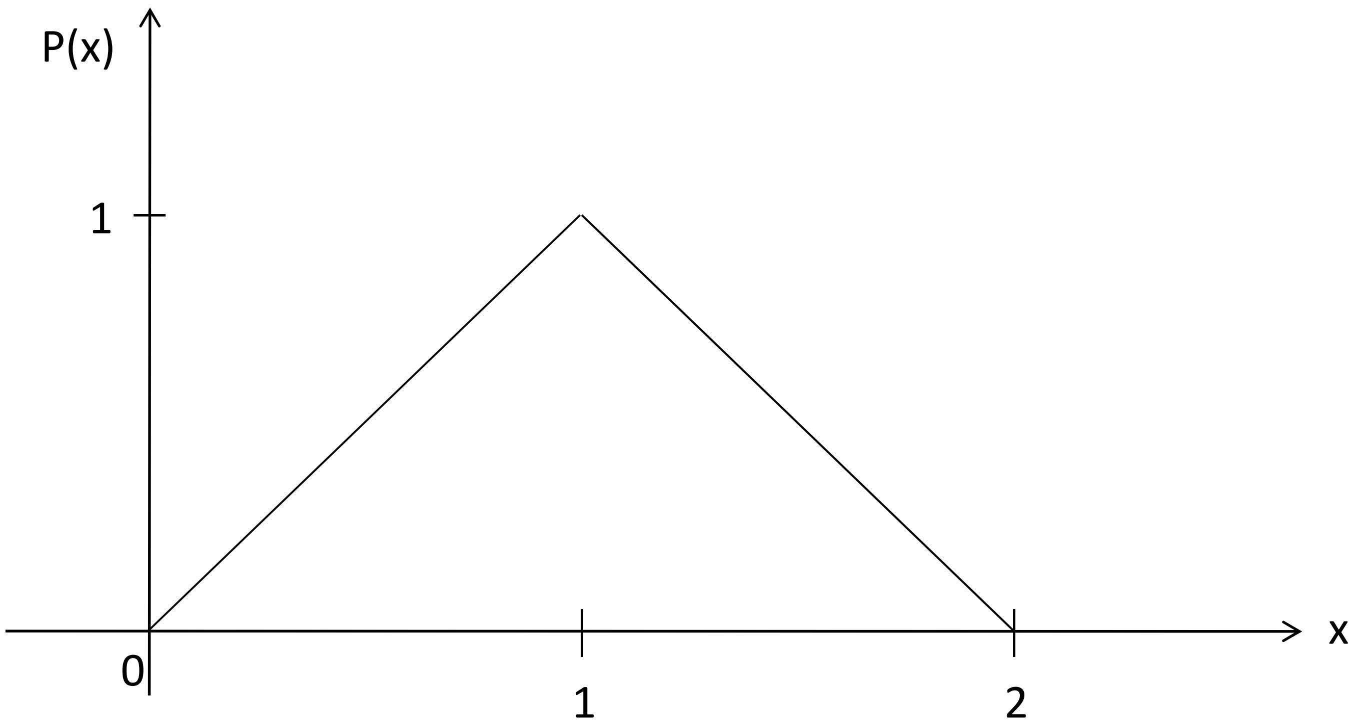
Example

For a fair dice, the probability distribution function is discrete for integers from 1 to 6:



Example

Consider probability density function:



Probability for x to occur between 0.5 and 0.51

$$\approx 0.5 \times 0.01 = 0.005$$

Normalization condition of the probability density function

If $x \in [x_a, x_b]$.

Discrete case:

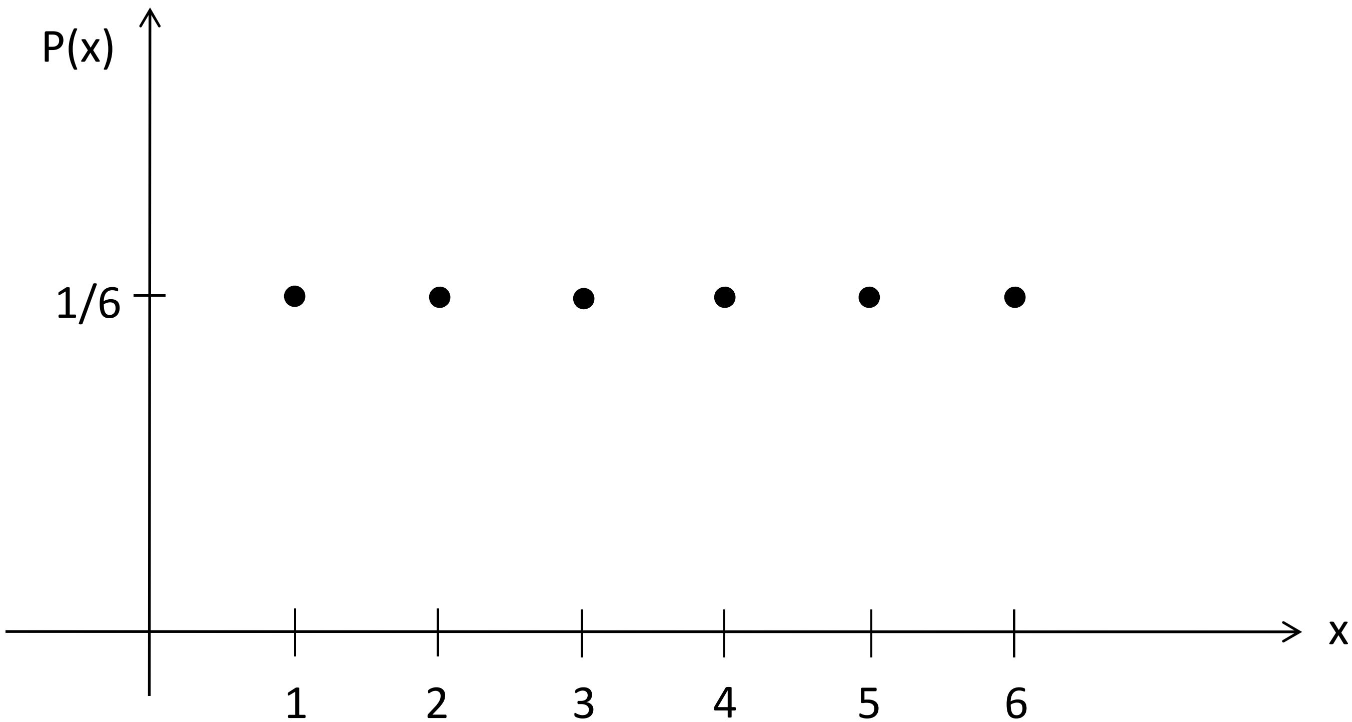
$$\sum_{x_i = x_a}^{x_i = x_b} p(x_i) = 1$$

Continuous case:

$$\int_{x_a}^{x_b} p(x)\, dx = 1$$

Example

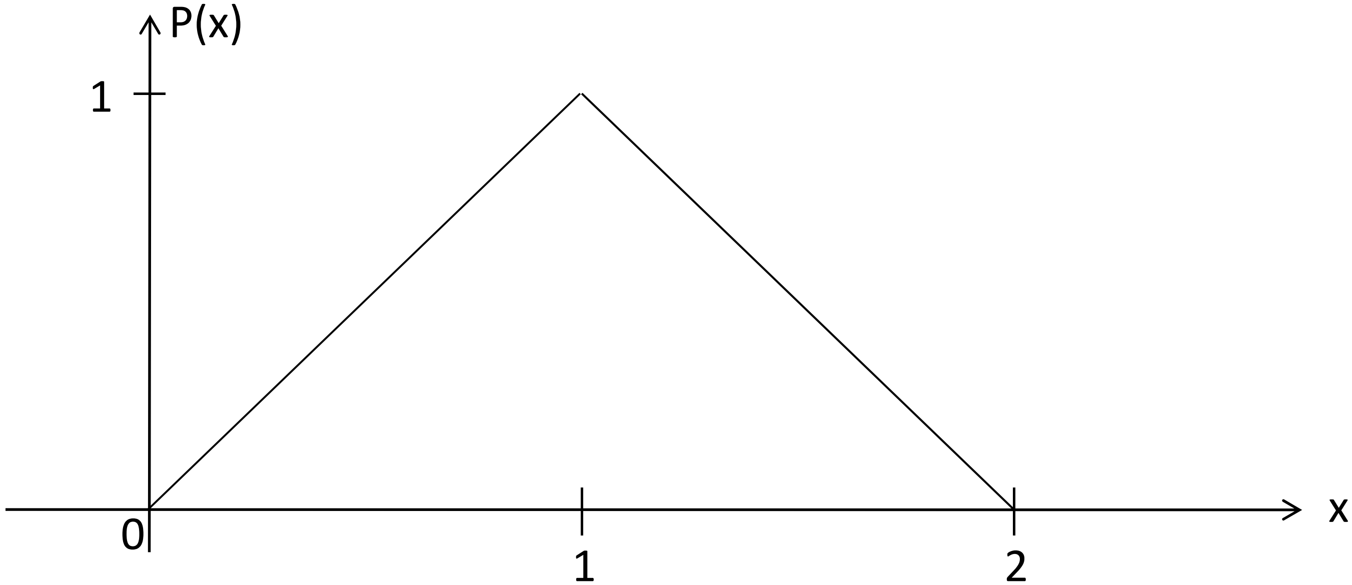
For a fair dice, the probability distribution function is discrete for integers from 1 to 6:



$$\sum_{x_i=1}^{6} p(x_i) = \frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=1$$

Example

Consider probability density function:



$$\int P(x)\,dx = \int_0^1 x\,dx + \int_1^2 (2-x)\,dx = \left[\frac{x^2}{2}\right]_0^1 + \left[2x - \frac{x^2}{2}\right]_1^2$$

$$= \left[\frac{1}{2} - 0\right] + \left[\left(4 - \frac{4}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} + \frac{4}{2} - \frac{3}{2} = 1$$

Example

A fair dice is thrown, if the out come is 1, I pay you \$5.9. Otherwise, you pay me \$1.00. What is the expectation of your earning?

x∈{1,2,3,4,5,6}. If f(x)= your earning,. f(1) = 5.9, f(2)=f(3)=f(4)=f(5)=f(6)=-1.0

Since the dice is fair, p(1)=p(2)=p(3)=p(4)=p(5)=p(6)=1/6

∴<f(x)>=\$5.9/6-\$1.0/6 -\$1.0/6 -\$1.0/6 -\$1.0/6 -\$1.0/6

=\$5.9/6-\$5.0/6

= \$0.15

∴ I expect you to earn \$0.15 per game.

Mean

The mean of a distribution p(x) is just the expectation of x, <x> or $\overline{x}$.

$$\text{If } x \in [x_a, x_b].$$

Discrete case:

$$< x > = \sum_{x_i = x_a}^{x_i = x_b} x_i p(x_i)$$

Continuous case:

$$< x > = \int_{x_a}^{x_b} x p(x)\, dx$$

Example

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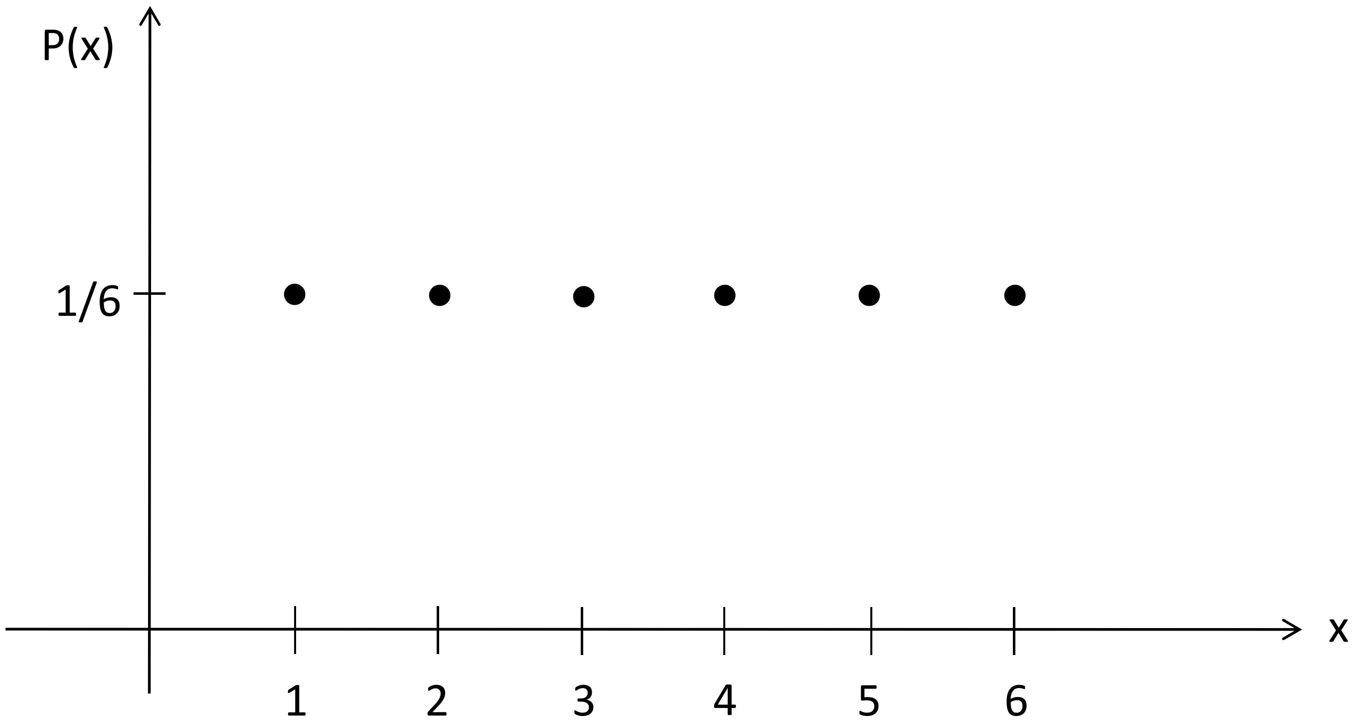


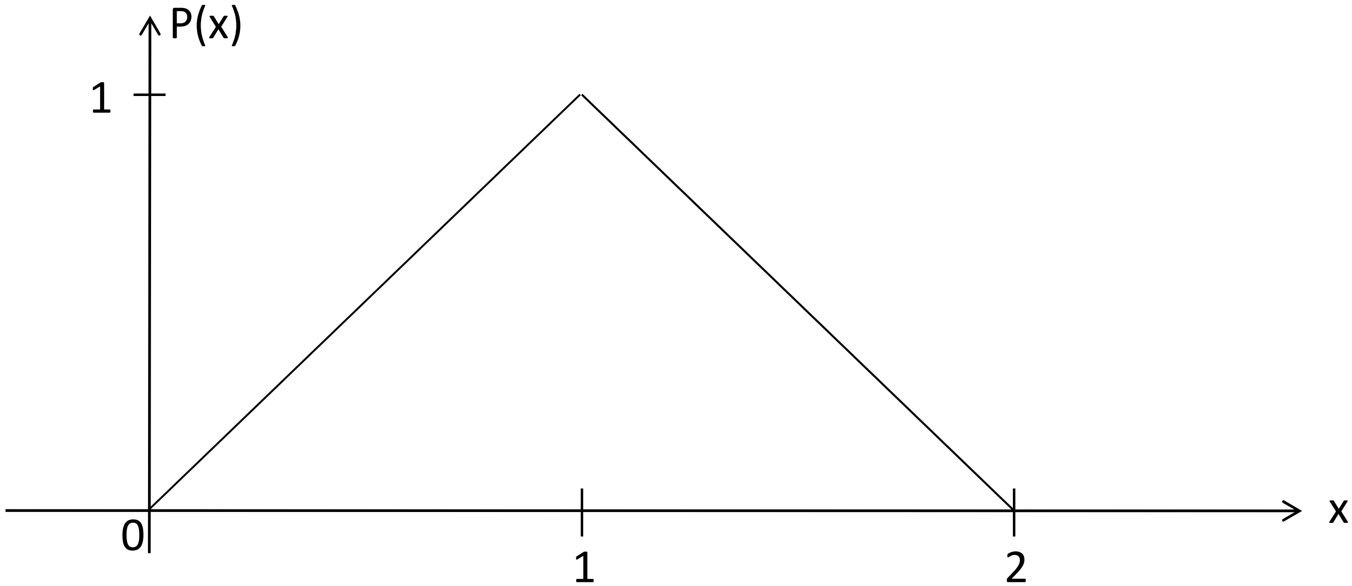
$$\text{Mean} < x > = \sum_{x_i=1}^{6} x_i p(x_i)$$

$$= \frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6} = \frac{21}{6} = 3.5$$

The mean is the average face value after many trials.

Example

Consider probability density function:



$$<x> = \int xP(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 x(2-x)\,dx = \left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2$$

$$= \left[\frac{1}{3} - 0\right] + \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] = \frac{1}{3} + \frac{4}{3} - \frac{2}{3} = 1$$