

Properties of wave functions (Text 5.1)

Classical mechanics turns out to be just an approximate version of quantum mechanics. The certainties of classical mechanics are illusory, and their apparent agreement with experiment occurs because ordinary objects consist of so many individual atoms that departures from average behavior are unnoticeable. Instead of two sets of physical principles, one for the macroworld and one for the microworld, there is only the single set included in quantum mechanics.

The quantity with which quantum mechanics is concerned is the wave function $\Psi(\vec{r}, t)$.

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1. $\Psi(\vec{r}, t)$ is complex. It can be written in the form

$$\Psi(\vec{r}, t) = A(\vec{r}, t) + i B(\vec{r}, t)$$

where A and B are real functions.

2. Complex conjugate of Ψ is defined as

$$\Psi^* = A - iB$$

3. $|\Psi|^2 = \Psi^* \Psi = A^2 + B^2$

Therefore $|\Psi|^2 = \Psi^* \Psi$ is always positive and real.

4. While Ψ itself has no physical interpretation, $|\Psi|^2$ evaluated at a particular place at a particular time equals to the probability of finding the body there at that time.

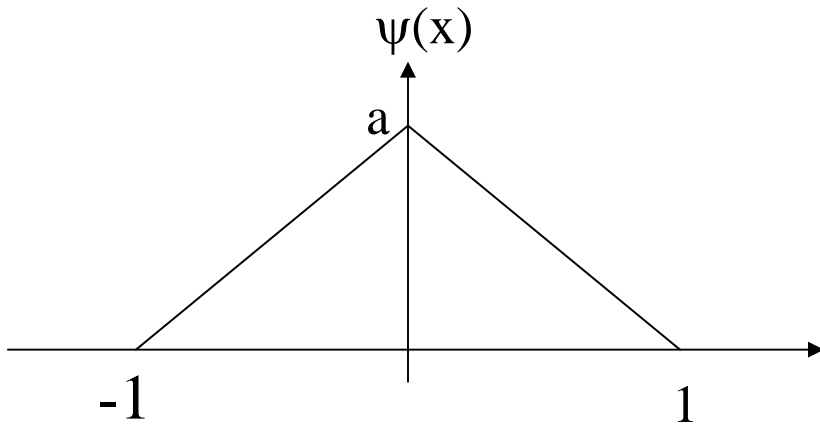
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5. Normalization

$$\int_{-\infty}^{\infty} |\Psi(x, t)|^2 dV = 1$$

If a wavefunction is not normalized, we can make it so by dividing it with a normalization constant. E.g.

$$f(x) = \begin{cases} a(1-x) & x \geq 0 \\ a(1+x) & x < 0 \end{cases}$$



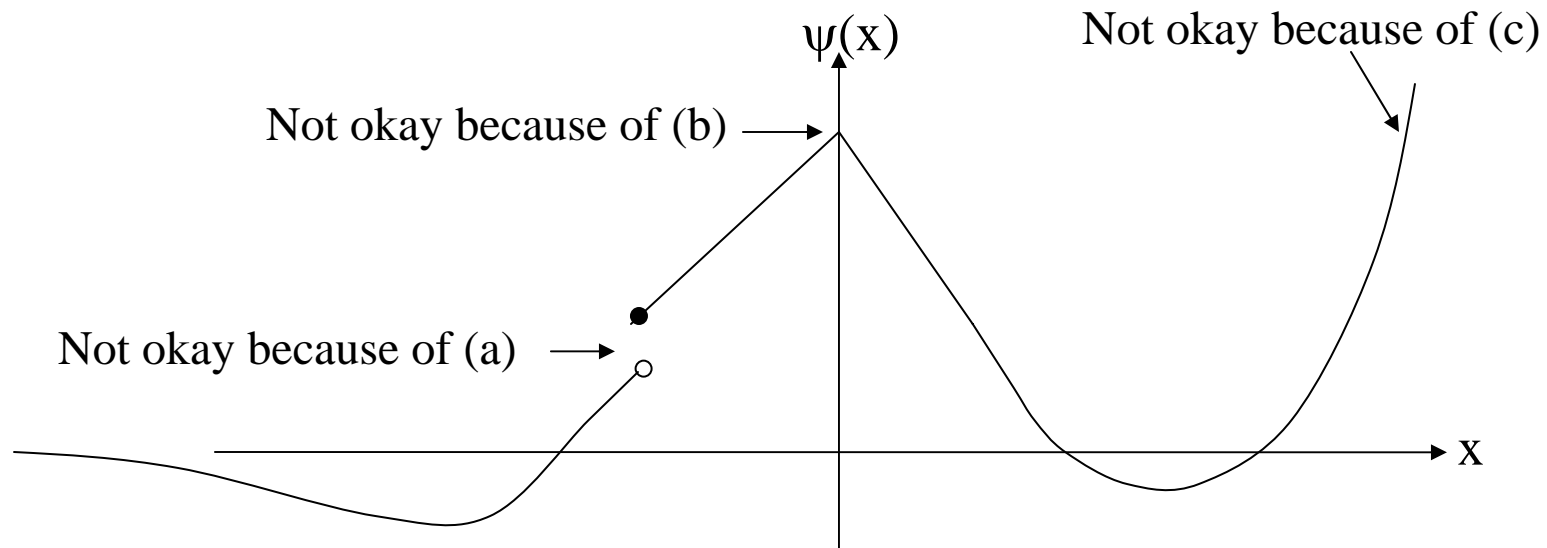
$$\begin{aligned} \therefore \int_{-\infty}^{\infty} |f(x)|^2 dx &= 2 \int_0^1 [a(1-x)]^2 dx \\ &= 2a^2 \left[-\frac{(1-x)^3}{3} \right]_0^1 \\ &= \frac{2}{3}a^2 \neq 1 \end{aligned}$$

$$\therefore f(x) \text{ is not normalised, but } \psi(x) = \frac{f(x)}{\sqrt{\frac{2}{3}a}} \text{ is !}$$

Properties of wave functions (Text 5.1)

5. Mathematical properties of Ψ :

- a. Ψ must be continuous and single-valued everywhere.
- b. $\partial\Psi/\partial x$, $\partial\Psi/\partial y$, $\partial\Psi/\partial z$ must be continuous and single-valued everywhere. (There may be exception in some special situations, we will discuss this later.)
- c. Ψ must be normalizable. $|\Psi|^2$ must go to 0 fast enough as x , y , or $z \rightarrow \pm\infty$ so that $\int |\Psi|^2 dV$ remains finite.



Expectation values (Text 5.5)

In probability, if the probability density function is $P(x)$, then the expectation of any function G of x is given by

$$\langle G \rangle = \int_{-\infty}^{\infty} P(x)G(x)dx$$

or in discrete case,

$$\langle G \rangle = \sum_i P_i G_i$$

In words, if you do the experiment, the probability of the outcome to occur in the range x to $x+dx$ is given by $P(x)dx$. With this outcome, $G(x)$ can be calculated. If you do the experiments many times and calculate the value of $G(x)$ every time. The average value of G can be predicted as $\langle G \rangle$ in the above equation.

Expectation values (Text 5.5)



Example. In a game you guess the outcome of a dice. If your guess is correct, you will be paid the amount same (in \$) as the value of the dice. What is the expectation of your earning?

$$\begin{aligned} \langle \$ \rangle &= \underbrace{\frac{1}{6} \times \$1}_{\text{Outcomeis 1}} + \underbrace{\frac{1}{6} \times \$2}_{\text{Outcomeis 2}} + \underbrace{\frac{1}{6} \times \$3}_{\text{Outcomeis 3}} + \underbrace{\frac{1}{6} \times \$4}_{\text{Outcomeis 4}} + \underbrace{\frac{1}{6} \times \$5}_{\text{Outcomeis 5}} + \underbrace{\frac{1}{6} \times \$6}_{\text{Outcomeis 6}} \\ &= \$3.5 \end{aligned}$$

This means if you play the game many times, you will earn \$3.5 per game on average. To be fair, you should be charged \$3.5 for each game (a cassino will charge you slightly more for cost and profit).

Expectation values (Text 5.5)

Since $|\Psi|^2$ is interpreted as a probability density function, so for any physical quantity G that is a function of x , we can calculate its expectation values as

$$\langle G \rangle = \int_{-\infty}^{\infty} |\psi(x)|^2 G(x) dx$$

However, the situation is different from previous example. In quantum mechanics, we cannot repeat the measurement many times and take $\langle G \rangle$ as the average, because each measurement will affect the wave function Ψ .

Expectation values (Text 5.5)

Example. A particle limited to the x axis has the wave function $\psi=ax^2+ibx$ between $x=0$ and $x=1$; $\psi=0$ elsewhere. Find the expectation value $\langle x \rangle$ of the particle's position. (See also example 5.2 of text)

$$\begin{aligned} |\psi(x)|^2 &= \psi^* \psi = (ax^2 + ibx)(ax^2 - ibx) \\ &= a^2 x^4 + b^2 x^2 \end{aligned}$$

$$\therefore \langle x \rangle = \int_0^1 x |\psi(x)|^2 dx$$

$$\Rightarrow \langle x \rangle = \int_0^1 x(a^2 x^4 + b^2 x^2) dx$$

$$\Rightarrow \langle x \rangle = \int_0^1 (a^2 x^5 + b^2 x^3) dx$$

$$\Rightarrow \langle x \rangle = \left[\frac{a^2}{6} x^6 + \frac{b^2}{4} x^4 \right]_0^1$$

$$\Rightarrow \langle x \rangle = \underline{\underline{\frac{a^2}{6} + \frac{b^2}{4}}}$$

Note that there is a relationship between a and b, to be determined by the normalization condition. Can you figure out the relationship?