Name:________________________________

University of Kentucky
Department of Physics and Astronomy

PHY 520 Introduction to Quantum Mechanics
Fall 2002
Final Examination

Answer all questions. Write down all work in detail.
Time allowed: 120 minutes
Merry Christmas and Happy New Year!

1. A particle of mass m is confined to a one dimensional region $0 \le x \le a$ with the potential

$$V(x) \;=\; \begin{cases} \infty & x < 0 \\ 0 & 0 < x < a \\ \infty & x > 0 \end{cases}$$

At t=0 its normalized wave function is

$$\Psi(x, t = 0) = \sqrt{\frac{8}{5a}}\left[1 + \cos\left(\frac{\pi x}{a}\right)\right]\sin\left(\frac{\pi x}{a}\right)$$

(a) (8 points)
What is the wave function at a later time $t = t_0$? The Schroedinger eigenfunctions and eigenvalues for the above potential are given as

$$\Psi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad ; \; E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$\Psi(x,t=0)=\sqrt{\frac{8}{5a}}\left[1+\cos\left(\frac{\pi x}{a}\right)\right]\sin\left(\frac{\pi x}{a}\right)$$

$$=\sqrt{\frac{8}{5a}}\sin\left(\frac{\pi x}{a}\right)+\sqrt{\frac{2}{5a}}\sin\left(\frac{2\pi x}{a}\right)=\frac{2}{\sqrt{5}}\Psi_1+\frac{1}{\sqrt{5}}\Psi_2$$

$$\therefore \Psi(x,t)\ =\ \frac{2}{\sqrt{5}}\Psi_1 e^{-i(E_1/\hbar)t}+\frac{1}{\sqrt{5}}\Psi_2 e^{-i(E_2/\hbar)t}$$

$$=\frac{2}{\sqrt{5}}\Psi_1\exp\left(\frac{-i\pi^2\hbar t_0}{2ma^2}\right)+\frac{1}{\sqrt{5}}\Psi_2\exp\left(\frac{-i2\pi^2\hbar t_0}{ma^2}\right)$$

$$=\underline{\underline{\sqrt{\frac{8}{5a}}\sin\left(\frac{\pi x}{a}\right)\exp\left(\frac{-i\pi^2\hbar t_0}{2ma^2}\right)+\sqrt{\frac{2}{5a}}\sin\left(\frac{2\pi x}{a}\right)\exp\left(\frac{-i2\pi^2\hbar t_0}{ma^2}\right)}}$$

$$\text{or}\quad \underline{\underline{\sqrt{\frac{8}{5a}}\left[1+\cos\left(\frac{\pi x}{a}\right)\exp\left(\frac{-i3\pi^2\hbar t_0}{2ma^2}\right)\right]\exp\left(\frac{-i\pi^2\hbar t_0}{2ma^2}\right)\sin\left(\frac{\pi x}{a}\right)}}$$

(b) (9 points)
What is the average energy of the syatem at t=0 and at t=t_0?

$$\therefore \Psi(x,t)\ =\ \frac{2}{\sqrt{5}}\Psi_1 e^{-i(E_1/\hbar)t}+\frac{1}{\sqrt{5}}\Psi_2 e^{-i(E_2/\hbar)t}$$

$$<E>=\int\left[\frac{2}{\sqrt{5}}\Psi_1^* e^{i(E_1/\hbar)t}+\frac{1}{\sqrt{5}}\Psi_2^* e^{i(E_2/\hbar)t}\right]\hat{H}\left[\frac{2}{\sqrt{5}}\Psi_1 e^{-i(E_1/\hbar)t}+\frac{1}{\sqrt{5}}\Psi_2 e^{-i(E_2/\hbar)t}\right]dx$$

$$=\int\left[\frac{2}{\sqrt{5}}\Psi_1^* e^{i(E_1/\hbar)t}+\frac{1}{\sqrt{5}}\Psi_2^* e^{i(E_2/\hbar)t}\right]\left[\frac{2E_1}{\sqrt{5}}\Psi_1 e^{-i(E_1/\hbar)t}+\frac{E_2}{\sqrt{5}}\Psi_2 e^{-i(E_2/\hbar)t}\right]dx$$

$$=\frac{4}{5}E_1+\frac{1}{5}E_2$$

$$=\frac{4}{5}\frac{\pi^2\hbar^2}{2ma^2}+\frac{1}{5}\frac{4\pi^2\hbar^2}{2ma^2}$$

$$=\underline{\underline{\frac{4}{5}\frac{\pi^2\hbar^2}{ma^2}}}$$

This is a constant of motion and has the same value ofr $t = 0$ and $t = t_0$.

c. (8 points)

What is the probability that the particle is found in the left half of the box (i.e., in the region $0 \le x \le a/2$) at $t=t_0$?

$$P(0 \le x \le a/2) = \int_0^{a/2} \left|\Psi(x,t_0)\right|^2 dx$$

$$= \int_0^{a/2} \left| \sqrt{\frac{8}{5a}} \sin\left(\frac{\pi x}{a}\right) \exp\left(\frac{-i\pi^2\hbar t_0}{2ma^2}\right) + \sqrt{\frac{2}{5a}} \sin\left(\frac{2\pi x}{a}\right) \exp\left(\frac{-i2\pi^2\hbar t_0}{ma^2}\right) \right|^2 dx$$

$$= \int_0^{a/2} \left\{ \left[\sqrt{\frac{8}{5a}} \sin\left(\frac{\pi x}{a}\right)\right]^2 + \left[\sqrt{\frac{2}{5a}} \sin\left(\frac{2\pi x}{a}\right)\right]^2 \right\} dx$$

$$+ \int_0^{a/2} 2\sqrt{\frac{8}{5a}} \sin\left(\frac{\pi x}{a}\right) \sqrt{\frac{2}{5a}} \sin\left(\frac{2\pi x}{a}\right) \cos\frac{3\pi^2\hbar t_0}{2ma^2} dx$$

$$= \frac{8}{5a} \int_0^{a/2} \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi x}{a}\right)\right] dx + \frac{2}{5a} \int_0^{a/2} \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{4\pi x}{a}\right)\right] dx$$

$$+ \frac{4}{5a} \cos\left(\frac{3\pi^2\hbar t_0}{2ma^2}\right) \int_0^{a/2} \left\{ \cos\left[\left(\frac{2\pi x}{a}\right) - \left(\frac{\pi x}{a}\right)\right] - \cos\left[\left(\frac{2\pi x}{a}\right) + \left(\frac{\pi x}{a}\right)\right] \right\} dx$$

$$= \frac{8}{5a}\left\{\frac{a}{4}\right\} + \frac{2}{5a}\left\{\frac{a}{4}\right\} + \frac{4}{5a} \cos\left(\frac{3\pi^2\hbar t_0}{2ma^2}\right) \int_0^{a/2} \left\{ \cos\left(\frac{\pi x}{a}\right) - \cos\left(\frac{3\pi x}{a}\right) \right\} dx$$

$$= \frac{1}{2} + \frac{4}{5a} \cos\left(\frac{3\pi^2\hbar t_0}{2ma^2}\right) \left\{ \frac{a}{\pi}\left[\sin\left(\frac{\pi x}{a}\right)\right]_0^{a/2} - \frac{a}{3\pi}\left[\sin\left(\frac{3\pi x}{a}\right)\right]_0^{a/2} \right\}$$

$$= \frac{1}{2} + \frac{4}{5a} \cos\left(\frac{3\pi^2\hbar t_0}{2ma^2}\right) \left\{ \frac{a}{\pi} + \frac{a}{3\pi} \right\}$$

$$= \underline{\underline{\frac{1}{2} + \frac{16}{15\pi} \cos\left(\frac{3\pi^2\hbar t_0}{2ma^2}\right)}}$$

2. The time independent normalized wave function for an electron in a hydrogen atom is

$$\psi(\vec{r}) = \frac{1}{\sqrt{10}}(2\psi_{100} + \psi_{210} + \sqrt{2}\psi_{211} + \sqrt{3}\psi_{21-1})$$

where the subscripts are values of the quantum number *n*, *l*, *m*.

(a) (7 points)
Calculate $<L^2>$ and $<L_z>$ for the electron.

$$< \psi(\vec{r}) \,|\, L^2 \,|\, \psi(\vec{r}) >$$

$$= \frac{1}{10}(4 < \psi_{100} \,|\, L^2 \,|\, \psi_{100} > + < \psi_{210} \,|\, L^2 \,|\, \psi_{210} > +$$

$$2 < \psi_{211} \,|\, L^2 \,|\, \psi_{211} > +3 < \psi_{21-1} \,|\, L^2 \,|_{21-1}>)$$

$$= \frac{1}{10}[4 \cdot 0(0+1)\hbar^2 + 1 \cdot 1(1+1)\hbar^2 + 2 \cdot 1(1+1)\hbar^2 + 3 \cdot 1(1+1)\hbar^2]$$

$$= \frac{12}{10}\hbar^2 = \underline{\underline{\frac{6}{5}\hbar^2}}$$

$$< \psi(\vec{r}) \,|\, L_z \,|\, \psi(\vec{r}) >$$

$$= \frac{1}{10}(4 < \psi_{100} \,|\, L_z \,|\, \psi_{100} > + < \psi_{210} \,|\, L_z \,|\, \psi_{210} > +$$

$$2 < \psi_{211} \,|\, L_z \,|\, \psi_{211} > +3 < \psi_{21-1} \,|\, L_z \,|_{21-1}>)$$

$$= \frac{1}{10}[4 \cdot 0\hbar + 1 \cdot 0\hbar + 2 \cdot 1\hbar + 3 \cdot (-1)\hbar]$$

$$= \underline{\underline{-\frac{1}{10}\hbar}}$$

(b) (7 points)

The first few radial functions R_{nl} for hydrogen atom is given as

$$R_{10}(r) = 2a_0^{-3/2}e^{-r/a_0}$$

$$R_{20}(r) = 2(2a_0)^{-3/2}\left(1 - \frac{r}{2a_0}\right)e^{-r/2a_0}$$

$$R_{21}(r) = \frac{(2a_0)^{-3/2}}{\sqrt{3}}\frac{r}{a_0}e^{-r/2a_0}$$

where a_0 is the Bohr radius, a constant.

Calculate the probability of finding the electron within a distance R from the proton. You can assume $R \ll a_0$.

$$P(r < R) = \int \psi^*(\vec{r})\,\psi(\vec{r})\,d^3r$$

$$= \frac{1}{10}\int_0^R (4R_{10}^2 + R_{21}^2 + 2R_{21}^2 + 3R_{21}^2)r^2dr$$

$$= \frac{1}{10}\int_0^R (4R_{10}^2 + 6R_{21}^2)r^2dr$$

$$R_{10}^2(r) = 4a_0^{-3}e^{-2r/a_0} \approx \frac{4}{a_0^3}\left(1 - \frac{2r}{a_0}\right)$$

$$R_{21}^2(r) = \frac{(2a_0)^{-3}}{3}\left(\frac{r}{a_0}\right)^2 e^{-r/a_0} = \frac{r^2}{24a_0^5}e^{-r/a_0} \approx \frac{r^2}{24a_0^5}\left(1 - \frac{r}{a_0}\right)$$

$$\therefore P = \frac{1}{10}\int_0^R \left[4\cdot\frac{4}{a_0^3}\left(1 - \frac{2r}{a_0}\right) + 6\cdot\frac{r^2}{24a_0^5}\left(1 - \frac{r}{a_0}\right)\right]r^2dr$$

$$\approx \frac{1}{10}\int_0^R 4\cdot\frac{4}{a_0^3}r^2dr \qquad \text{(taking lowest order of r)}$$

$$= \frac{16}{10}\left[\frac{r^2}{3a_0^3}\right]_0^R = \underline{\underline{\frac{8}{15}\left(\frac{R}{a_0}\right)^3}}$$

(c) (4 points)

Multiple choices. Circle the correct answer.

The ground state energy of hydrogen atom is

A. 0 eV
[✓ B. -13.6 eV]
C. -0.53 eV
D. 13.6 eV
E. 0.53 eV

How does E_n depend on n:

A. $E_n \sim n^2$
B. $E_n \sim n$
C. $E_n \sim 1/n$
[✓ D. $E_n \sim 1/n^2$]
E. $E_n \sim n\hbar\omega$

(d) (7 points)

Calculate <E> for the electron. (Hint: use results from part (c) above).

$$< E > = < \psi(\bar{r}) | H | \psi(\bar{r}) >$$

$$= \frac{1}{10}(4 < \psi_{100} | H | \psi_{100} > + < \psi_{210} | H | \psi_{210} > +$$

$$2 < \psi_{211} | H | \psi_{211} > + 3 < \psi_{21-1} | H |_{21-1} >)$$

$$= \frac{1}{10}[4E_1 + 1E_2 + 2E_2 + 3E_2]$$

$$= \frac{1}{5}[2E_1 + 3E_2]$$

$$= \frac{1}{5}[2 \times (-13.6\text{eV}) + 3 \times (\frac{-13.6\text{eV}}{2^2})]$$

$$= \frac{11}{20} \times (-13.6\text{eV}) = \underline{\underline{-7.48\text{eV}}}$$

3. A particle in a central potential has an orbital angular momentum L=2← and a spin S = 1←. The spin-orbit interaction gives rise to the Hamiltonian H= A **L•S** where A is a constant. Let **J** = **L**+**S**.

(a) (5 points)
Altogether how many eigenstates are there?

$L_z = +2, +1, 0, -1, -5.$ 5 possible values.

$S_z = +1, 0, -1.$ 3 possible values.

$\therefore$ We expect a total of $5 \times 3 = 15$ eigenstate.

(b) (5 points)
What are the quantum numbers that determine the eigenstates of the Hamiltonian?

L_z and S_z are not good quantum numbers for $\vec{L} \cdot \vec{S}$.

The good quantum numbers are now ℓ, s, j, and j_z.

(c) (5 points)

Write the Hamiltonian in terms of J^2, L^2, and S^2.

$$\vec{J} = \vec{L} + \vec{S} \;\Rightarrow\; J^2 = L^2 + S^2 - 2\vec{L}\cdot\vec{S} \;\Rightarrow \vec{L}\cdot\vec{S} = \frac{1}{2}\left(J^2 - L^2 - S^2\right)$$

$$\Rightarrow H = A\vec{L}\cdot\vec{S} = \underline{\underline{\frac{A}{2}\left(J^2 - L^2 - S^2\right)}}$$

(d) (10 points)

Determine all possible energy eigenvalues. Determine the corresponding degeneracy for each of these eigenvalues.

$j = |\ell - s|, |\ell - s| + 1, ...0, ..., \ell + s - 1, \ell + s$

$= 1, 2, 3.$

For $j = 1$:

$E = \frac{A}{2}[1(1+1) - 2(2+1) - 1(1+1)]\hbar^2 = -3A\hbar^2$

Possible j_z : $-1, 0, 1$

$\therefore$ Degeneracy $= 3$

For $j = 2$:

$E = \frac{A}{2}[2(2+1) - 2(2+1) - 1(1+1)]\hbar^2 = -A\hbar^2$

Possible j_z : $-2, -1, 0, 1, 2$

$\therefore$ Degeneracy $= 5$

For $j = 3$:

$E = \frac{A}{2}[3(3+1) - 2(2+1) - 1(1+1)]\hbar^2 = 2A\hbar^2$

Possible j_z : $-3, -2, -1, 0, 1, 2$

$\therefore$ Degeneracy $= 7$

$\therefore$ Total number of states $= 3 + 5 + 7 = 15$, same as part (a).