University of Kentucky
Department of Physics and Astronomy

PHY 520 Introduction to Quantum Mechanics
Fall 2002
Test 1

Answer all questions. Write down all work in detail.
Time allowed: 50 minutes
Good luck!

1. Consider a wave packet given by

$$f(x) = \begin{cases} 0 & x < -a/2 \\ A & -a/2 < x < a/2 \\ 0 & x > a/2 \end{cases}$$

(a) (3 POINTS)
Find the value of A to normalize f(x).

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 1 \quad \Rightarrow \int_{-a/2}^{a/2} A^2 dx = 1$$

$$\Rightarrow A^2 a = 1$$

$$\Rightarrow A = \frac{1}{\sqrt{a}}$$

(b) (12 points)
Find the form of g(k). Make sure g(k) is normalized also.

$$g(k) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-ikx} f(x) dx = \int_{-a/2}^{a/2} \frac{1}{\sqrt{2\pi}} e^{-ikx} \cdot \frac{1}{\sqrt{a}} dx$$

$$= \frac{1}{\sqrt{2\pi a}} \int_{-a/2}^{a/2} e^{-ikx} dx$$

$$= \frac{1}{\sqrt{2\pi a}} \left[\frac{e^{-ikx}}{-ik} \right]_{-a/2}^{a/2}$$

$$= \frac{1}{\sqrt{2\pi a}} \frac{e^{-ika/2} - e^{ika/2}}{-ik}$$

$$= \frac{2}{\sqrt{2\pi a}} \frac{\sin ka/2}{k}$$

$$= \sqrt{\frac{2}{\pi a}} \frac{\sin ka/2}{k}$$

(c) (3 points)

Write down f(x,t) as a series expansion of $e^{i(kx-\omega t)}$.

$$f(x,t) = \sum_{k=-\infty}^{\infty} g(k) e^{i(kx-\omega t)} = \underline{\underline{\sum_{k=-\infty}^{\infty} \sqrt{\frac{2}{\pi a}} \frac{\sin ka/2}{k} e^{i(kx-\omega t)}}}$$

(d) (7 points)

Show that a reasonable definition of Δk for you answer to (b) yields

$$\Delta x \, \Delta k > 1$$

independent of the value of a.

Peak of $|g(0)|^2$ occurs at k = 0 with peak value given as

$$|g(0)|^2 = \frac{2}{\pi a}\left(\frac{a}{2}\right)^2 = \frac{a}{2\pi}$$

Value of k that gives rise to half of this value can be estimated as

$$\frac{2}{\pi a}\frac{1}{k^2} = \frac{1}{2}\frac{a}{2\pi} \Rightarrow \frac{1}{k^2} = \frac{a^2}{8} \Rightarrow k = \frac{\sqrt{8}}{a} = \frac{2\sqrt{2}}{a}$$

$$\therefore \Delta k = 2 \times \frac{2\sqrt{2}}{a} = \frac{4\sqrt{2}}{a}$$

It is obvious that $\Delta x = a$

$$\therefore \Delta x \Delta k = a \cdot \frac{4\sqrt{2}}{a} = 4\sqrt{2} > 1, \text{ independent of the value of a.}$$

2. A particle of mass m is confined to a one dimensional region $0 \le x \le a$ with the potential

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 < x < a \\ \infty & x > 0 \end{cases}$$

At t=0 its normalized wave function is

$$\Psi(x, t=0) = \sqrt{\frac{8}{5a}}\left[1+\cos\left(\frac{\pi x}{a}\right)\right]\sin\left(\frac{\pi x}{a}\right)$$

(a) (8 points)
What is the wave function at a later time $t = t_0$? The Schroedinger eigenfunctions and eigenvalues for the above potential are given as

$$\Psi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right) \qquad ; \; E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

$$\Psi(x, t=0) = \sqrt{\frac{8}{5a}}\left[1+\cos\left(\frac{\pi x}{a}\right)\right]\sin\left(\frac{\pi x}{a}\right)$$

$$= \sqrt{\frac{8}{5a}}\sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{2}{5a}}\sin\left(\frac{2\pi x}{a}\right) = \frac{2}{\sqrt{5}}\Psi_1 + \frac{1}{\sqrt{5}}\Psi_2$$

$$\therefore \Psi(x,t) = \frac{2}{\sqrt{5}}\Psi_1 e^{-i(E_1/\hbar)t} + \frac{1}{\sqrt{5}}\Psi_2 e^{-i(E_2/\hbar)t}$$

$$= \frac{2}{\sqrt{5}}\Psi_1 \exp\left(\frac{-i\pi^2\hbar t_0}{2ma^2}\right) + \frac{1}{\sqrt{5}}\Psi_2 \exp\left(\frac{-i2\pi^2\hbar t_0}{ma^2}\right)$$

$$= \underline{\underline{\sqrt{\frac{8}{5a}}\sin\left(\frac{\pi x}{a}\right)\exp\left(\frac{-i\pi^2\hbar t_0}{2ma^2}\right) + \sqrt{\frac{2}{5a}}\sin\left(\frac{2\pi x}{a}\right)\exp\left(\frac{-i2\pi^2\hbar t_0}{ma^2}\right)}}$$

or $$\underline{\underline{\sqrt{\frac{8}{5a}}\left[1+\cos\left(\frac{\pi x}{a}\right)\exp\left(\frac{-i3\pi^2\hbar t_0}{2ma^2}\right)\right]\exp\left(\frac{-i\pi^2\hbar t_0}{2ma^2}\right)\sin\left(\frac{\pi x}{a}\right)}}$$

(b) (9 points)

What is the average energy of the syatem at t=0 and at t=t_0?

$$\therefore \Psi(x,t) = \frac{2}{\sqrt{5}}\Psi_1 e^{-i(E_1/\hbar)t} + \frac{1}{\sqrt{5}}\Psi_2 e^{-i(E_2/\hbar)t}$$

$$< E >= \int\left[\frac{2}{\sqrt{5}}\Psi_1^* e^{i(E_1/\hbar)t} + \frac{1}{\sqrt{5}}\Psi_2^* e^{i(E_2/\hbar)t}\right]\hat{H}\left[\frac{2}{\sqrt{5}}\Psi_1 e^{-i(E_1/\hbar)t} + \frac{1}{\sqrt{5}}\Psi_2 e^{-i(E_2/\hbar)t}\right]dx$$

$$= \int\left[\frac{2}{\sqrt{5}}\Psi_1^* e^{i(E_1/\hbar)t} + \frac{1}{\sqrt{5}}\Psi_2^* e^{i(E_2/\hbar)t}\right]\left[\frac{2E_1}{\sqrt{5}}\Psi_1 e^{-i(E_1/\hbar)t} + \frac{E_2}{\sqrt{5}}\Psi_2 e^{-i(E_2/\hbar)t}\right]dx$$

$$= \frac{4}{5}E_1 + \frac{1}{5}E_2$$

$$= \frac{4}{5}\frac{\pi^2\hbar^2}{2ma^2} + \frac{1}{5}\frac{4\pi^2\hbar^2}{2ma^2}$$

$$= \frac{4}{5}\frac{\pi^2\hbar^2}{ma^2}$$

This is a constant of motion and has the same value ofr $t = 0$ and $t = t_0$.

c. (8 points)

What is the probability that the particle is found in the left half of the box (i.e., in the region $0 \le x \le a/2$) at t=t_0?

See next page.

$$P(0 \le x \le a/2) = \int_0^{a/2} \left|\Psi(x,t_0)\right|^2 dx$$

$$= \int_0^{a/2} \left| \sqrt{\frac{8}{5a}} \sin\left(\frac{\pi x}{a}\right) \exp\left(\frac{-i\pi^2 \hbar t_0}{2ma^2}\right) + \sqrt{\frac{2}{5a}} \sin\left(\frac{2\pi x}{a}\right) \exp\left(\frac{-i2\pi^2 \hbar t_0}{ma^2}\right) \right|^2 dx$$

$$= \int_0^{a/2} \left\{ \left[\sqrt{\frac{8}{5a}} \sin\left(\frac{\pi x}{a}\right) \right]^2 + \left[\sqrt{\frac{2}{5a}} \sin\left(\frac{2\pi x}{a}\right) \right]^2 \right\} dx$$

$$+ \int_0^{a/2} 2\sqrt{\frac{8}{5a}} \sin\left(\frac{\pi x}{a}\right) \sqrt{\frac{2}{5a}} \sin\left(\frac{2\pi x}{a}\right) \cos\frac{3\pi^2 \hbar t_0}{2ma^2} dx$$

$$= \frac{8}{5a} \int_0^{a/2} \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi x}{a}\right) \right] dx + \frac{2}{5a} \int_0^{a/2} \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{4\pi x}{a}\right) \right] dx$$

$$+ \frac{4}{5a} \cos\left(\frac{3\pi^2 \hbar t_0}{2ma^2}\right) \int_0^{a/2} \left\{ \cos\left[\left(\frac{2\pi x}{a}\right) - \left(\frac{\pi x}{a}\right) \right] - \cos\left[\left(\frac{2\pi x}{a}\right) + \left(\frac{\pi x}{a}\right) \right] \right\} dx$$

$$= \frac{8}{5a}\left\{\frac{a}{4}\right\} + \frac{2}{5a}\left\{\frac{a}{4}\right\} + \frac{4}{5a} \cos\left(\frac{3\pi^2 \hbar t_0}{2ma^2}\right) \int_0^{a/2} \left\{ \cos\left(\frac{\pi x}{a}\right) - \cos\left(\frac{3\pi x}{a}\right) \right\} dx$$

$$= \frac{1}{2} + \frac{4}{5a} \cos\left(\frac{3\pi^2 \hbar t_0}{2ma^2}\right) \left\{ \frac{a}{\pi} \left[\sin\left(\frac{\pi x}{a}\right) \right]_0^{a/2} - \frac{a}{3\pi} \left[\sin\left(\frac{3\pi x}{a}\right) \right]_0^{a/2} \right\}$$

$$= \frac{1}{2} + \frac{4}{5a} \cos\left(\frac{3\pi^2 \hbar t_0}{2ma^2}\right) \left\{ \frac{a}{\pi} + \frac{a}{3\pi} \right\}$$

$$= \frac{1}{2} + \frac{16}{15\pi} \cos\left(\frac{3\pi^2 \hbar t_0}{2ma^2}\right)$$