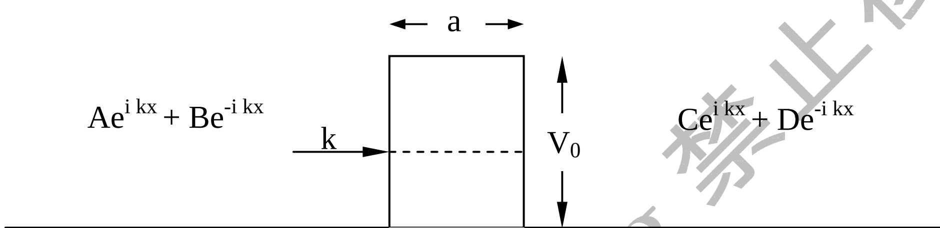
University of Kentucky
Department of Physics and Astronomy

PHY 520 Introduction to Quantum Mechanics
Fall 2002
Test 2

Answer all questions. Write down all work in detail.
Time allowed: 50 minutes
Show all work clearly. Add extra papers if needed. Good luck!

1. Consider the finite potential barrier shown in the figure:



The scattering matrix is given as:

$$\begin{pmatrix} C \\ B \end{pmatrix} = S \begin{pmatrix} A \\ D \end{pmatrix}$$

$$S = \frac{e^{-2ika}}{\cosh 2\kappa a + \frac{i\varepsilon}{2}\sinh 2\kappa a} \begin{pmatrix} 1 & -\frac{i\eta}{2}\sinh 2\kappa a \\ -\frac{i\eta}{2}\sinh 2\kappa a & 1 \end{pmatrix}$$

where

$$\varepsilon = \frac{\kappa}{k} - \frac{k}{\kappa} \text{ and } \eta = \frac{\kappa}{k} + \frac{k}{\kappa}; \quad \frac{\hbar^2 k^2}{2m} = E \text{ and } \frac{\hbar^2 \kappa^2}{2m} = V_0 - E$$

(a) (5 points)
Is S unitary, or Hermitian, or both, or none? Prove your answer.

S is unitary.

$S^{+}S$

$$= \frac{e^{2ika}}{\cosh 2\kappa a - \frac{i\varepsilon}{2}\sinh 2\kappa a}\begin{pmatrix} 1 & \frac{i\eta}{2}\sinh 2\kappa a \\ \frac{i\eta}{2}\sinh 2\kappa a & 1 \end{pmatrix}$$

$$\times \frac{e^{-2ika}}{\cosh 2\kappa a + \frac{i\varepsilon}{2}\sinh 2\kappa a}\begin{pmatrix} 1 & -\frac{i\eta}{2}\sinh 2\kappa a \\ -\frac{i\eta}{2}\sinh 2\kappa a & 1 \end{pmatrix}$$

$$= \frac{1}{\cosh^2 2\kappa a + \frac{\varepsilon^2}{4}\sinh^2 2\kappa a}\begin{pmatrix} 1+\frac{\eta^2}{4}\sinh^2 2\kappa a & 0 \\ 0 & 1+\frac{\eta^2}{4}\sinh^2 2\kappa a \end{pmatrix}$$

$$= \frac{1}{\cosh^2 2\kappa a + \frac{\sinh^2 2\kappa a}{4}\left(\frac{\kappa^2}{k^2}+2+\frac{k^2}{\kappa^2}\right)}\begin{pmatrix} 1+\frac{\eta^2}{4}\sinh^2 2\kappa a & 0 \\ 0 & 1+\frac{\eta^2}{4}\sinh^2 2\kappa a \end{pmatrix}$$

$$= \frac{1}{(1-\sinh^2 2\kappa a) + \frac{\sinh^2 2\kappa a}{4}\left(\frac{\kappa^2}{k^2}+2+\frac{k^2}{\kappa^2}\right)}\begin{pmatrix} 1+\frac{\eta^2}{4}\sinh^2 2\kappa a & 0 \\ 0 & 1+\frac{\eta^2}{4}\sinh^2 2\kappa a \end{pmatrix}$$

$$= \frac{1}{1 + \frac{\sinh^2 2\kappa a}{4}\left(\frac{\kappa^2}{k^2}+2+\frac{k^2}{\kappa^2}\right)}\begin{pmatrix} 1+\frac{\eta^2}{4}\sinh^2 2\kappa a & 0 \\ 0 & 1+\frac{\eta^2}{4}\sinh^2 2\kappa a \end{pmatrix}$$

$$= \frac{1}{1 + \frac{\sinh^2 2\kappa a}{4}\eta^2}\begin{pmatrix} 1+\frac{\eta^2}{4}\sinh^2 2\kappa a & 0 \\ 0 & 1+\frac{\eta^2}{4}\sinh^2 2\kappa a \end{pmatrix}$$

$$= I$$

It is obvious that S^{+} does not equal to S, so S is not Hermitian.

(b) (10 points)

Calculate the transmission probability $|T|^2$. Further simplify your answer by assuming $|T| << 1$.

$$\begin{pmatrix} C \\ B \end{pmatrix} = \frac{e^{-2ika}}{\cosh 2\kappa a + \frac{i\varepsilon}{2}\sinh 2\kappa a} \begin{pmatrix} 1 & -\frac{i\eta}{2}\sinh 2\kappa a \\ -\frac{i\eta}{2}\sinh 2\kappa a & 1 \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix}$$

With $D = 0$ and $A = 1$, then $C = T$.

i.e. $$T = \frac{e^{-2ika}}{\cosh 2\kappa a + \frac{i\varepsilon}{2}\sinh 2\kappa a} \Rightarrow |T|^2 = \frac{1}{\cosh^2 2\kappa a + \frac{\varepsilon^2}{4}\sinh^2 2\kappa a}$$

$$\begin{pmatrix} C \\ B \end{pmatrix} = \frac{e^{-2ika}}{\cosh 2\kappa a + \frac{i\varepsilon}{2}\sinh 2\kappa a} \begin{pmatrix} 1 & -\frac{i\eta}{2}\sinh 2\kappa a \\ -\frac{i\eta}{2}\sinh 2\kappa a & 1 \end{pmatrix} \begin{pmatrix} A \\ D \end{pmatrix}$$

With $D = 0$ and $A = 1$, then $C = T$.

i.e. $$T = \frac{e^{-2ika}}{\cosh 2\kappa a + \frac{i\varepsilon}{2}\sinh 2\kappa a} \Rightarrow |T|^2 = \frac{1}{\cosh^2 2\kappa a + \frac{\varepsilon^2}{4}\sinh^2 2\kappa a}$$

With $T << 1$, $\cosh 2\kappa a \approx \sinh 2\kappa a \approx \frac{\exp(2\kappa a)}{2}$

$$|T|^2 \approx \frac{1}{\left(1+\frac{\varepsilon^2}{4}\right)\left(\frac{\exp(2\kappa a)}{2}\right)^2} \approx \frac{4}{\left\{1+\frac{1}{4}\left[\left(\frac{\kappa}{k}\right)^2 + \left(\frac{k}{\kappa}\right)^2 - 2\right]\right\}\exp(4\kappa a)}$$

$$= \frac{4}{\frac{1}{4}\left[\left(\frac{\kappa}{k}\right)^2 + \left(\frac{k}{\kappa}\right)^2 + 2\right]\exp(4\kappa a)}$$

$$= \frac{16}{\left[\left(\frac{\kappa}{k}\right) + \left(\frac{k}{\kappa}\right)\right]^2}\exp(-4\kappa a)$$

$$= \frac{16\kappa^2 k^2}{\left[\kappa^2 + k^2\right]^2}\exp(-4\kappa a)$$

$$= \frac{16E(V_0 - E)}{\left[E + (V_0 - E)\right]^2}\exp(-4\kappa a)$$

$$= \frac{16E(V_0 - E)}{V_0^2}\exp(-4\kappa a)$$

(c) (10 points)

An electron with energy E= 1eV is incident upon the potential barrier with V_0 = 2eV. About how wide must the barrier be so that the transmission probability is (i) 10^{-3}, (ii) 10^{-2} ? It is given the mass and charge of electrons are 9.11×10^{-27}g and 1.6×10^{-19} C respectively. Also, $\hbar = 1.055\times10^{-27}$ erg - s.

We define the barrier width a as shown in above figure.

$$E = 1\text{eV} = 1.6\times10^{-19}\,\text{J} = 1.6\times10^{-12}\,\text{erg}$$

$$V_0 = 2\text{eV} = 3.2\times10^{-12}\,\text{erg}$$

$$\kappa = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} = \sqrt{\frac{2\times9.11\times10^{-27}\times(3.2\times10^{-12} - 1.6\times10^{-12})}{(1.055\times10^{-27})^2}} = 1.62\times10^{8}\,\text{cm}^{-1}$$

$$T^2 = \frac{16E(V_0 - E)}{V_0^{\;2}}\exp(-4\kappa a) = \frac{16\times1\times(2-1)}{2^2}\exp(-4\kappa a)$$

$$\Rightarrow 4\kappa a = \ell n\left(\frac{4}{T^2}\right) \Rightarrow a = \frac{1}{4\kappa}\ell n\left(\frac{4}{T^2}\right) = \frac{1}{4\times1.62\times10^{8}}\ell n\left(\frac{4}{T^2}\right)$$

$$= 1.54\times10^{-9}\,\ell n\left(\frac{4}{T^2}\right)$$

(i) With $T^2 = 0.001$, $\ell n\left(\frac{4}{0.001}\right) = 8.29$

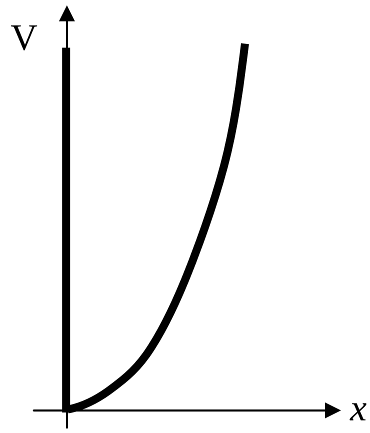
$$a = 1.54\times10^{-9}\times8.29 = \underline{\underline{1.28\times10^{-8}\,\text{cm}}}$$

(ii) With $T^2 = 0.01$, $\ell n\left(\frac{4}{0.01}\right) = 5.99$

$$a = 1.54\times10^{-9}\times5.99 = \underline{\underline{9.26\times10^{-9}\,\text{cm}}}$$

There is only a small difference in barrier width, but it has caused a 10 fold decrease in T^2.

2. Consider a semi-simple harmonic potential as shown in the figure:



$$V(x)=\begin{cases}\infty & x<0\\ \frac{1}{2}kx^2 & x\geq 0\end{cases}$$

(a) (7 points)

Which of the following CANNOT be an eigenfunction (for $x \geq 0$)of this potential? Explain your reasoning carefully. In all these equations, $y = (2\pi m\omega/h)^{1/2}$ and $\omega=(k/m)^{1/2}$. A is a kind of normalization constant to be determined.

(i) $\Psi(x) = A(32y^5 - 160y^3 + 129y)\exp(y^2/2)$

(ii) $\Psi(x) = A(16y^4 - 48y^2 + 12)\exp(-y^2/2)$

(ii) $\Psi(x) = A(32y^5 + 160y^3 + 129y)\exp(-y^2/2)$

(iv) $\Psi(x) = A\exp(-y^2/2)$

(v) $\Psi(x) = A(2y)\ \exp(-y^2/2)$

(vi) $\Psi(x) = A(8y^3 + 4y^2 - 12y)\exp(-y^2/2)$

(vii) $\Psi(x) = A(32y^5 - 160y^3 + 129y)$

(i) Not possible. Wave function is nor square integrable.
(ii) Not possible. Wave function is not zero at origin
(iii) Not possible. Hermite polynomial should have alternate sign.
(iv) Not possible. Wave function is not zero at origin
(v) Possible.
(vi) Not possible. Wave function is not odd.
(vii) Not possible. Wave function is nor square integrable.

(b) (8 points)

What are the energy eigenvalues for this potential? What is the gound state energy?

The solution of the Schroedinger equation for this potential is essentially the same as the simple harmonic oscillator, except that the wave function has to be 0 at the origin. Therefore, we take only the odd solution for simple harmonic oscillator, i.e. n=1, 3, 5, 7,….
The energy eigenvalues for this potential are:

$$E_n = (n + \tfrac{1}{2})\hbar\omega, \quad n = 1,3,5,7,....$$

Ground state energy is $E = \frac{3}{2}\hbar\omega$.

(c) (10 points)
In all possible eigengunctions from part (a), choose one to calculate the normalization constant A, and the position expectation <x>. What is the energy of this eigenfunction? The following integrals may be useful:

$$\int_0^\infty e^{-t^2} t^{2z-1} dt = \frac{\Gamma(z)}{2} \text{ ; and } \Gamma(n) = (n-1)\Gamma(n-1); \text{ and } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

There is only one possible eigenfunction :

$$\Psi(x) = A\,[\exp(-y^2/2)](2y)\,,\ y = \sqrt{\frac{m\omega}{\hbar}}x$$

For normalization :

$$\int_0^\infty \Psi^* \Psi dx = 1 \quad \Rightarrow \int_0^\infty A^2 \exp(-y^2)(4y^2)\, dx = 1$$

$$\Rightarrow \sqrt{\frac{\hbar}{m\omega}} \int_0^\infty 4A^2 y^2 \exp(-y^2) dy = 1$$

$$\Rightarrow 4\sqrt{\frac{\hbar}{m\omega}} A^2 \frac{\Gamma(\frac{3}{2})}{2} = 1$$

$$\Rightarrow 2A^2 (\frac{1}{2}) \left(\frac{\hbar\pi}{m\omega}\right)^{1/2} = 1$$

$$\Rightarrow A^2 \left(\frac{h}{2m\omega}\right)^{1/2} = 1$$

$$\Rightarrow A = \left(\frac{2m\omega}{h}\right)^{1/4}$$

$$
\begin{aligned}
< x > &= \int_0^\infty \Psi^* x \Psi dx \\
&= \left(\frac{2m\omega}{h}\right)^{1/2} \int_0^\infty \exp(-y^2)(4y^2) x dx \\
&= \left(\frac{2m\omega}{h}\right)^{1/2} \int_0^\infty \exp(-y^2)(4y^2) \sqrt{\frac{\hbar}{m\omega}} y \sqrt{\frac{\hbar}{m\omega}} dy \\
&= 4\left(\frac{2m\omega}{h}\right)^{1/2} \frac{\hbar}{m\omega} \int_0^\infty \exp(-y^2)(y^3) dy \\
&= 4\left(\frac{2m\omega}{h}\right)^{1/2} \frac{\hbar}{m\omega} \frac{\Gamma(2)}{2} \\
&= 2\left(\frac{2m\omega}{h}\right)^{1/2} \frac{\hbar}{m\omega} \\
&= \left(4 \times \frac{2m\omega}{h} \times \frac{h^2}{4\pi^2 m^2 \omega^2}\right)^{1/2} \\
&= \underline{\underline{\left(\frac{2h}{\pi^2 m\omega}\right)^{1/2}}}
\end{aligned}
$$