University of Kentucky
Department of Physics and Astronomy

PHY 520 Introduction to Quantum Mechanics
Fall 2003
Final Examination

Date: Dec 15, 2003
Time: 8:00-10:00
Answer all questions.

1. (Total: 40 points)
A particle of mass m is confined to a one dimensional region $0 \le x \le a$ with the potential

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 < x < a \\ \infty & x > 0 \end{cases}$$

At t=0 its normalized wave function is

$$\Psi(x, t = 0) = \sqrt{\frac{8}{5a}}\left[1 + \cos\left(\frac{\pi x}{a}\right)\right]\sin\left(\frac{\pi x}{a}\right)$$

(a) (6 points)
Write down all the energy eigenstates $\psi_n(x)$ and the corresponding energy E_n.

$$\Psi_n = \sqrt{\frac{2}{a}}\sin kx = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$$
$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{n^2\pi^2\hbar^2}{2ma^2}$$

(b) (10 points)

What is the wave function at a later time $t = t_0$?

$$\Psi(x,t=0)=\sqrt{\frac{8}{5a}}\left[1+\cos\left(\frac{\pi x}{a}\right)\right]\sin\left(\frac{\pi x}{a}\right)$$

$$=\sqrt{\frac{8}{5a}}\sin\left(\frac{\pi x}{a}\right)+\sqrt{\frac{2}{5a}}\sin\left(\frac{2\pi x}{a}\right)=\frac{2}{\sqrt{5}}\Psi_1+\frac{1}{\sqrt{5}}\Psi_2$$

$$\therefore\ \Psi(x,t)=\frac{2}{\sqrt{5}}\Psi_1 e^{-i(E_1/\hbar)t}+\frac{1}{\sqrt{5}}\Psi_2 e^{-i(E_2/\hbar)t}$$

$$=\frac{2}{\sqrt{5}}\Psi_1\exp\left(\frac{-i\pi^2\hbar t_0}{2ma^2}\right)+\frac{1}{\sqrt{5}}\Psi_2\exp\left(\frac{-i2\pi^2\hbar t_0}{ma^2}\right)$$

$$=\sqrt{\frac{8}{5a}}\sin\left(\frac{\pi x}{a}\right)\exp\left(\frac{-i\pi^2\hbar t_0}{2ma^2}\right)+\sqrt{\frac{2}{5a}}\sin\left(\frac{2\pi x}{a}\right)\exp\left(\frac{-i2\pi^2\hbar t_0}{ma^2}\right)$$

or

$$\sqrt{\frac{8}{5a}}\left[1+\cos\left(\frac{\pi x}{a}\right)\exp\left(\frac{-i3\pi^2\hbar t_0}{2ma^2}\right)\right]\exp\left(\frac{-i\pi^2\hbar t_0}{2ma^2}\right)\sin\left(\frac{\pi x}{a}\right)$$

(c) (10 points)

What is the expectation energy of the system at $t=0$ and at $t=t_0$?

$$\therefore\ \Psi(x,t)=\frac{2}{\sqrt{5}}\Psi_1 e^{-i(E_1/\hbar)t}+\frac{1}{\sqrt{5}}\Psi_2 e^{-i(E_2/\hbar)t}$$

$$<E>=\int\left[\frac{2}{\sqrt{5}}\Psi_1^* e^{i(E_1/\hbar)t}+\frac{1}{\sqrt{5}}\Psi_2^* e^{i(E_2/\hbar)t}\right]\hat{H}\left[\frac{2}{\sqrt{5}}\Psi_1 e^{-i(E_1/\hbar)t}+\frac{1}{\sqrt{5}}\Psi_2 e^{-i(E_2/\hbar)t}\right]dx$$

$$=\int\left[\frac{2}{\sqrt{5}}\Psi_1^* e^{i(E_1/\hbar)t}+\frac{1}{\sqrt{5}}\Psi_2^* e^{i(E_2/\hbar)t}\right]\left[\frac{2E_1}{\sqrt{5}}\Psi_1 e^{-i(E_1/\hbar)t}+\frac{E_2}{\sqrt{5}}\Psi_2 e^{-i(E_2/\hbar)t}\right]dx$$

$$=\frac{4}{5}E_1+\frac{1}{5}E_2$$

$$=\frac{4}{5}\frac{\pi^2\hbar^2}{2ma^2}+\frac{1}{5}\frac{4\pi^2\hbar^2}{2ma^2}$$

$$=\frac{4}{5}\frac{\pi^2\hbar^2}{ma^2}$$

This is a constant of motion and has the same value ofr $t = 0$ and $t = t_0$.

(d) (14 points)

What is the probability that the particle is found in the left half of the box (i.e., in the region $0 \le x \le a/2$) at $t=t_0$?

$$P(0 \le x \le a/2) = \int_0^{a/2} \left|\Psi(x,t_0)\right|^2 dx$$

$$= \int_0^{a/2} \left| \sqrt{\frac{8}{5a}} \sin\left(\frac{\pi x}{a}\right) \exp\left(\frac{-i\pi^2\hbar t_0}{2ma^2}\right) + \sqrt{\frac{2}{5a}} \sin\left(\frac{2\pi x}{a}\right) \exp\left(\frac{-i2\pi^2\hbar t_0}{ma^2}\right) \right|^2 dx$$

$$= \int_0^{a/2} \left\{ \left[\sqrt{\frac{8}{5a}} \sin\left(\frac{\pi x}{a}\right)\right]^2 + \left[\sqrt{\frac{2}{5a}} \sin\left(\frac{2\pi x}{a}\right)\right]^2 \right\} dx$$

$$+ \int_0^{a/2} 2\sqrt{\frac{8}{5a}} \sin\left(\frac{\pi x}{a}\right) \sqrt{\frac{2}{5a}} \sin\left(\frac{2\pi x}{a}\right) \cos\frac{3\pi^2\hbar t_0}{2ma^2} dx$$

$$= \frac{8}{5a}\int_0^{a/2} \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi x}{a}\right)\right] dx + \frac{2}{5a}\int_0^{a/2} \left[\frac{1}{2} - \frac{1}{2}\cos\left(\frac{4\pi x}{a}\right)\right] dx$$

$$+ \frac{4}{5a}\cos\left(\frac{3\pi^2\hbar t_0}{2ma^2}\right) \int_0^{a/2} \left\{ \cos\left[\left(\frac{2\pi x}{a}\right) - \left(\frac{\pi x}{a}\right)\right] - \cos\left[\left(\frac{2\pi x}{a}\right) + \left(\frac{\pi x}{a}\right)\right] \right\} dx$$

$$= \frac{8}{5a}\left\{\frac{a}{4}\right\} + \frac{2}{5a}\left\{\frac{a}{4}\right\} + \frac{4}{5a}\cos\left(\frac{3\pi^2\hbar t_0}{2ma^2}\right) \int_0^{a/2} \left\{ \cos\left(\frac{\pi x}{a}\right) - \cos\left(\frac{3\pi x}{a}\right) \right\} dx$$

$$= \frac{1}{2} + \frac{4}{5a}\cos\left(\frac{3\pi^2\hbar t_0}{2ma^2}\right) \left\{ \frac{a}{\pi}\left[\sin\left(\frac{\pi x}{a}\right)\right]_0^{a/2} - \frac{a}{3\pi}\left[\sin\left(\frac{3\pi x}{a}\right)\right]_0^{a/2} \right\}$$

$$= \frac{1}{2} + \frac{4}{5a}\cos\left(\frac{3\pi^2\hbar t_0}{2ma^2}\right) \left\{ \frac{a}{\pi} + \frac{a}{3\pi} \right\}$$

$$= \frac{1}{2} + \frac{16}{15\pi}\cos\left(\frac{3\pi^2\hbar t_0}{2ma^2}\right)$$

2. (Total: 60 points)
For all parts of this problem, consider the case l =1.

(a) (7 points)
With z-axis as the rotational axis, fill in the following commutation relation:

$[L_x, L_y] = i\hbar L_z$

$[L_z, L_y] = -i\hbar L_x$

$[L^2, L_y] = 0$

$[L^2, L_z] = 0$

$[L_+, L_x] = -\hbar L_+$

$[L_+, L_z] = -\hbar L_+$

$[L_+, L_-] = 2\hbar L_z$

(b) (5 points)
With l = 1, how many eigenstates are there for L_z and what are their eigenvalues?

Three eigenstates, with m = -1, 0 and 1 and hence the eigenvalues are $-\hbar$, 0, and $\hbar$.

(c) (8 points)
With respect to the eigenstates of L_z, write down the operators L^2 and L_z in matrix form.

$$L^2 = \hbar \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(d) (8 points)

Write L_-L_+ in terms of L^2 and L_z for the case when $m \neq 1$. What should be the value of L_-L_+ if m=1?

$$L_\pm = L_x \pm iL_y \Rightarrow L_-L_+ = (L_x - iL_y)(L_x + iL_y) \quad \text{for } m \neq +1 \quad (= 0 \text{ if } m = +1)$$
$$= L_x{}^2 + L_y{}^2 + iL_xL_y - iL_yL_x$$
$$= L_x{}^2 + L_y{}^2 - \hbar L_z$$
$$= L^2 - L_z{}^2 - \hbar L_z$$

(e) (8 points)

Using your result in part (d) or otherwise, construct operators L_+ and L- in matrix form.

$$L_-L_+ = L^2 - L_z{}^2 - \hbar L_z = \hbar^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Note that the first element in the above matrice was "set" to 0 because $L_-L_+ = 0$ for $m = +1$ eigenstate.

Let $L_+ = \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}$ and $L- = \begin{pmatrix} 0 & 0 & 0 \\ \alpha^* & 0 & 0 \\ 0 & \beta^* & 0 \end{pmatrix}$ $\quad \therefore L_-L_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \beta^2 \end{pmatrix}$

$\therefore \alpha^2 = 2\hbar^2 \Rightarrow \alpha = \sqrt{2}\,\hbar$ and similarly $\beta = \sqrt{2}\hbar$

$\therefore L_+ = \hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}$ and $L_= = \hbar \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}$

(f) (8 points)
Construct operators L_x and L_y in matrix form from your results in part (d).

$$L_{\pm} = L_x \pm iL_y \Rightarrow L_x = \frac{1}{2}(L_+ + L_-) = \frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

$$L_y = \frac{1}{2i}(L_+ - L_-) = \frac{i}{2}(L_- - L_+) = \frac{i\hbar}{2}\begin{pmatrix} 0 & -\sqrt{2} & 0 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}$$

(g) (8 points)
What are the eigenvalues and eigenvectors of L_x?

L_x and L_y should have the same eigenvalues as that of L_z, i.e. $\hbar$, 0, and $-\hbar$.

Case 1. Eigenvalue $= \hbar$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \hbar\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}}y = x \\ \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}z = y \\ \frac{1}{\sqrt{2}}y = z \end{pmatrix} \Rightarrow x = z = \frac{1}{2}, y = \frac{1}{\sqrt{2}}$$

$$\therefore \text{The eigenvector is } \begin{pmatrix} \frac{1}{2} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$$

Case 2. Eigenvalue = 0

$$\frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \Rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}}y = 0 \\ \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}z = 0 \\ \frac{1}{\sqrt{2}}y = 0 \end{pmatrix} \Rightarrow x = \frac{1}{\sqrt{2}}, z = -\frac{1}{\sqrt{2}}, y = 0$$

$\therefore$ The eigenvector is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

Case 3. Eigenvalue = $-\hbar$

$$\frac{\hbar}{2}\begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\hbar\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}}y = -x \\ \frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}z = -y \\ \frac{1}{\sqrt{2}}y = -z \end{pmatrix} \Rightarrow x = z = \frac{1}{2}, y = -\frac{1}{\sqrt{2}}$$

$\therefore$ The eigenvector is $\begin{pmatrix} \frac{1}{2} \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}$

(h) (8 points)

Use the operators in matrix form to prove any one of the commutation relations in part (a).

Let us calculate $[L_+ , L_-]$:

$$\therefore L_+L_- = \hbar\begin{pmatrix}0 & \sqrt{2} & 0\\ 0 & 0 & \sqrt{2}\\ 0 & 0 & 0\end{pmatrix}\hbar\begin{pmatrix}0 & 0 & 0\\ \sqrt{2} & 0 & 0\\ 0 & \sqrt{2} & 0\end{pmatrix} = \hbar^2\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$\therefore L_-L_+ = \hbar\begin{pmatrix}0 & 0 & 0\\ \sqrt{2} & 0 & 0\\ 0 & \sqrt{2} & 0\end{pmatrix}\hbar\begin{pmatrix}0 & \sqrt{2} & 0\\ 0 & 0 & \sqrt{2}\\ 0 & 0 & 0\end{pmatrix} = \hbar^2\begin{pmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix}$$

$$[L_+ , L_-] = \hbar^2\begin{pmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 0\end{pmatrix} - \hbar^2\begin{pmatrix}0 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{pmatrix} = \hbar^2\begin{pmatrix}2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -2\end{pmatrix}$$

$$= 2\hbar\cdot\hbar\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

$$= 2\hbar L_z$$