University of Kentucky
Department of Physics and Astronomy

PHY 520 Introduction to Quantum Mechanics
Fall 2003
Test 1

Answer all questions. Write down all work in detail.
Time allowed: 50 minutes

A particle of mass m is confined to a one dimensional region $-a/2 \le x \le a/2$ with an infinite potential

$$V(x) = \begin{cases} \infty & x \le -\frac{a}{2} \\ 0 & -\frac{a}{2} < x < \frac{a}{2} \\ \infty & x \ge \frac{a}{2} \end{cases}$$

The wave function at t=0 is given as:

$$\Psi(x, t=0) = A\,\psi_1(x) + B\,\psi_2(x)$$

where $\psi_1(x)$ is the ground state and $\psi_2(x)$ is the first excited state.

Answer all of the following.

(a) (8 points)
Write down all the energy eigenstates $\psi_n(x)$ and the corresponding energy E_n.

$$\psi_n(x) = \begin{cases} \sqrt{\tfrac{2}{a}} \cos k_n x & \text{n odd} \\ \sqrt{\tfrac{2}{a}} \sin k_n x & \text{n even} \end{cases}$$

To satisfy boundary conditions, $k \cdot \frac{a}{2} = \frac{n\pi}{2} \Rightarrow k_n = \frac{n\pi}{a}$

$$\therefore \quad \psi_n(x) = \begin{cases} \sqrt{\frac{2}{a}} \cos \frac{n\pi x}{a} & \text{n odd} \\ \sqrt{\frac{2}{a}} \sin \frac{n\pi x}{a} & \text{n even} \end{cases}$$

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

(b) (8 points)
If the probability of finding the particle in the ground state is 2 times the probability of finding it in the first excited state, determine the value of A and B by assuming they are real numbers. Use these values for the remaining of this problem.

$$P\,(\text{ground state}) = A^2$$

$$P\,(\text{1st excited state}) = B^2$$

$$A^2 + B^2 = 1, \text{ but } A^2 = 2B^2 \quad \therefore 3B^2 = 1 \;\Rightarrow\; B = \sqrt{\frac{1}{3}} \;\text{ and } A = \sqrt{\frac{2}{3}}$$

(c) (10 points)
What is the probability that the particle is found in the right half of the box (i.e., in the region $0 \le x \le a/2$)?

$$P\,(0 \le x \le \tfrac{a}{2}) = \int_0^{\frac{a}{2}} \Psi^*(x)\Psi(x)dx$$

$$= \frac{2}{a}\int_0^{\frac{a}{2}} \left[\sqrt{\frac{2}{3}}\cos\frac{\pi x}{a} + \sqrt{\frac{1}{3}}\sin\frac{2\pi x}{a}\right]^2 dx$$

$$= \frac{2}{a}\int_0^{\frac{a}{2}} \left[\frac{2}{3}\cos^2\frac{\pi x}{a} + \frac{1}{3}\sin^2\frac{2\pi x}{a} + \frac{2\sqrt{2}}{3}\cos\frac{\pi x}{a}\sin\frac{2\pi x}{a}\right] dx$$

$$= \frac{2}{a}\int_0^{\frac{a}{2}} \left[\frac{2}{3}\left(\frac{1+\cos\frac{2\pi x}{a}}{2}\right) + \frac{1}{3}\left(\frac{1-\cos\frac{4\pi x}{a}}{2}\right) + \frac{2\sqrt{2}}{3}\,\frac{\sin\left(\frac{3\pi x}{a}\right) + \sin\left(\frac{\pi x}{a}\right)}{2}\right] dx$$

$$= \frac{2}{a}\left\{\left[\frac{1}{2}x\right]_0^{\frac{a}{2}} + \left[\frac{1}{3}\cdot\frac{a}{2\pi}\sin\frac{2\pi x}{a}\right]_0^{\frac{a}{2}} + \left[\frac{1}{6}\cdot\frac{a}{4\pi}\cos\frac{4\pi x}{a}\right]_0^{\frac{a}{2}} - \left[\frac{\sqrt{2}}{3}\frac{a}{3\pi}\cos\frac{3\pi x}{a}\right]_0^{\frac{a}{2}} - \left[\frac{\sqrt{2}}{3}\frac{a}{\pi}\cos\frac{\pi x}{a}\right]_0^{\frac{a}{2}}\right\}$$

$$= \frac{2}{a}\left\{\frac{a}{4} + 0 + 0 - \left(-\frac{\sqrt{2}}{3}\frac{a}{3\pi}\right) - \left(-\frac{\sqrt{2}}{3}\frac{a}{\pi}\right)\right\} = \underline{\underline{\frac{1}{2} + \frac{8\sqrt{2}}{9\pi}}}$$

(d) (10 points)
What is the probability of finding the particle in ground state ψ_1 at time t? How about the first excited state ψ_2? Do they depend on time?

At any time,

$$A(t) = A(0)e^{-iE_1/\hbar t}\ ;\ B(t) = B(0)e^{-iE_2/\hbar t}$$

$$|A(t)|^2 = \left|A(0)e^{-iE_1/\hbar t}\right|^2 = |A(0)|^2 = \frac{2}{3}$$

$$|B(t)|^2 = \left|B(0)e^{-iE_1/\hbar t}\right|^2 = |B(0)|^2 = \frac{1}{3}$$

They are constant over time.

(e) (8 points)
What is the probability of finding the particle in the left half of the box (i.e., in the region $0 \le x \le a/2$) at time t? Hint: Make use of your work in part (c), no need to do all integrations again!

$$\Psi(x,t) = \psi_1 e^{-i\omega_1 t} + \psi_2 e^{-i\omega_2 t}$$

$$\Rightarrow \Psi^*(x,t)\,\Psi(x,t) = (\psi_1^* e^{i\omega_1 t} + \psi_2^* e^{i\omega_2 t})\,(\psi_1 e^{-i\omega_1 t} + \psi_2 e^{-i\omega_2 t})$$

$$\Rightarrow \Psi^*(x,t)\,\Psi(x,t) = (\psi_1^*\psi_1 + \psi_2^*\psi_2 + \psi_1^*\psi_2 e^{i(\omega_1-\omega_2)t} + \psi_1\psi_2^* e^{i(\omega_2-\omega_1)t})$$

$$\Rightarrow \Psi^*(x,t)\,\Psi(x,t) = (\psi_1^2 + \psi_2^2 + 2\psi_1\psi_2 \cos(\omega_1-\omega_2)t$$

$$\therefore P(0 \le x \le a) = \int \Psi^*(x,t)\,\Psi(x,t)\,dx$$

$$= \int \psi_1^2 dx + \int \psi_2^2 dx + \cos(\omega_1-\omega_2)t \int 2\psi_1\psi_2 dx$$

Trace back to our integration in part (c),

$$\int \Psi^*(x,0)\,\Psi(x,0)\,dx = \underbrace{\frac{1}{2}}_{\int \psi_1^2 dx + \int \psi_2^2 dx} + \underbrace{\frac{8\sqrt{2}}{9\pi}}_{\int 2\psi_1\psi_2 dx}$$

$$\therefore P(0 \le x \le a) \text{ at any time} = \frac{1}{2} + \frac{8\sqrt{2}}{9\pi}\cos(\omega_1-\omega_2)t$$

(f) (8 points)
Is there any operator X that commute with the Hamiltonian of this problem? If so, calculate the expectation <X> of the state function $\Psi(x)$. Is <X> a constant over time? Briefly explain your answer.

Yes, the potential is symmetric about x=0 in this problem. Parity operator defined as Pf(x)=f(-x) will commute with H.

$$< P > = \Psi^+ P \Psi = (A^* \psi_1^{\ +} + B^* \psi_2^{\ +}) P (A\psi_1 + B\psi_2) = (A^* \psi_1^{\ +} + B^* \psi_2^{\ +})(A\psi_1 - B\psi_2)$$

$$= A^* A - B^* B \qquad (\psi_i^{\ +} \psi_j = \delta_{ij})$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$$

This will be constant overtime, since A^*A and B^*B are time independent in above calsulation. Also, $< X >$ is time independent if X commute with H.