University of Kentucky
Department of Physics and Astronomy

PHY 520 Introduction to Quantum Mechanics
Fall 2003
Test 2

Answer all questions. Write down all work in detail.
Time allowed: 50 minutes

Consider a double delta function potential

$$V(x) \;=\; -\frac{\hbar^2\lambda}{2ma}[\delta(x-a) + \delta(x+a)]$$

and E < 0.

Answer all of the following.

(a) (10 points)
Substitute the potential into the Schroedinger equation and simplify it.
Also write down the boundary conditions at x = a and x = -a.

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi(x)+V(x)\Psi(x)=E\,\Psi(x) \Rightarrow -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\Psi(x)+\left[-\frac{\hbar^2\lambda}{2ma}[\delta(x-a)+\delta(x+a)]\right]\Psi(x)=-|E|\,\Psi(x)$$

$$\Rightarrow -\frac{d^2}{dx^2}\Psi(x)-\frac{\lambda}{a}\left[\delta(x-a)+\delta(x+a)\right]\Psi(x)=-\frac{2m|E|}{\hbar^2}\Psi(x)$$

$$\Rightarrow \frac{d^2}{dx^2}\Psi(x)-\frac{2m|E|}{\hbar^2}\Psi(x) = -\frac{\lambda}{a}\left[\delta(x-a)+\delta(x+a)\right]\Psi(x)$$

$$\Rightarrow \frac{d^2}{dx^2}\Psi(x)-\kappa^2\;\Psi(x) = -\frac{\lambda}{a}\left[\delta(x-a)+\delta(x+a)\right]\Psi(x)$$

where $\kappa^2 = \frac{2m|E|}{\hbar^2}$.

Boundary conditions at x = a : $\Psi(a-\varepsilon)=\Psi(a+\varepsilon)$ (Condinuity of Ψ)

$$\Psi'(a+\varepsilon)-\Psi'(a-\varepsilon)=-\frac{\lambda}{a}\Psi(a)$$ (Discontinuity of first derivative at the delta function).

Similar conditions apply to $x=-a$.

(b) (10 points)
The solution to the Schroedinger equation has to be either even or odd. Explain why.

It is because the potential is symmetric on x, i.e. V(x)=V(-x).

(c) (20 points)
Solve the Schroedinger equation up to a normalization constant and show that

$$\tanh \kappa a = \frac{\lambda}{\kappa a} - 1 \qquad \text{for even solution}$$

$$\coth \kappa a = \frac{\lambda}{\kappa a} - 1 \qquad \text{for odd solution}$$

where $\kappa = \sqrt{\frac{2m|E|}{\hbar^2}}$

Describe the wavefunction in three regions separately, namely: $\Psi_I(x)$ for $-\infty < x < -a$, $\Psi_{II}(x)$ for $-a < x < a$, and $\Psi_{III}(x)$ for $-\Psi_I(x)$ for $a < x < \infty$.

If Ψ is even:

$$\Psi_I(x) = Ae^{\kappa x}, \quad \Psi_{II}(x) = B\cosh \kappa x, \quad \Psi_{III}(x) = Ae^{-\kappa x}$$

Continutily of $\Psi(x)$ at $x = a$ and $x = -a \Rightarrow Ae^{-\kappa a} - B\cosh \kappa a = 0$

Discontinuity of $\Psi(x)$ at $x = a$ and $x = -a \Rightarrow \delta\Psi'(a) = -\frac{\lambda}{a}\Psi(a) \Rightarrow -A\kappa e^{-\kappa a} - B\kappa \sinh \kappa a = -\frac{\lambda}{a} A\, e^{-\kappa a}$

$$\Rightarrow A\left(1 - \frac{\lambda}{\kappa a}\right)e^{-\kappa a} + B \sinh \kappa a = 0$$

For non-trivial solution in A and B,

$$\begin{vmatrix} e^{-\kappa a} & -\cosh \kappa a \\ \left(1 - \frac{\lambda}{\kappa a}\right)e^{-\kappa a} & \sinh \kappa a \end{vmatrix} = 0 \Rightarrow e^{-\kappa a} \sinh \kappa a + \left(1 - \frac{\lambda}{\kappa a}\right)e^{-\kappa a}\cosh \kappa a = 0 \Rightarrow \underline{\underline{\tanh \kappa a = \frac{\lambda}{\kappa a} - 1}}$$

If this condition is satisfied, A can be written in terms of B as $\underline{\underline{A = Be^{\kappa a}\cosh \kappa a}}$.

B has to be determined by normalization.

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For non-trivial solution in A and B,

$$\begin{vmatrix} e^{-\kappa a} & -\cosh \kappa a \\ \left(1 - \frac{\lambda}{\kappa a}\right)e^{-\kappa a} & \sinh \kappa a \end{vmatrix} = 0 \Rightarrow e^{-\kappa a}\sinh \kappa a + \left(1 - \frac{\lambda}{\kappa a}\right)e^{-\kappa a}\cosh \kappa a = 0 \Rightarrow \underline{\underline{\tanh \kappa a = \frac{\lambda}{\kappa a} - 1}}$$

If this condition is satisfied, A can be written in terms of B as $\underline{\underline{A = Be^{\kappa a}\cosh \kappa a}}$.

B has to be determined by normalization.

If Ψ is odd :

$$\Psi_I(x) = Ae^{\kappa x}, \quad \Psi_{II}(x) = B\sinh \kappa x, \quad \Psi_{III}(x) = -Ae^{-\kappa x}$$

Continutily of $\Psi(x)$ at $x = a$ and $x = -a \Rightarrow Ae^{-\kappa a} - B\sinh \kappa a = 0$

Discontinuity of $\Psi(x)$ at $x = a$ and $x = -a \Rightarrow \delta\Psi'(a) = -\frac{\lambda}{a}\Psi(a) \Rightarrow -A\kappa e^{-\kappa a} - B\kappa \cosh \kappa a = -\frac{\lambda}{a}A\, e^{-\kappa a}$

$$\Rightarrow A\left(1 - \frac{\lambda}{\kappa a}\right)e^{-\kappa a} + B\cosh \kappa a = 0$$

For non-trivial solution in A and B,

$$\begin{vmatrix} e^{-\kappa a} & -\sinh \kappa a \\ \left(1 - \frac{\lambda}{\kappa a}\right)e^{-\kappa a} & \cosh \kappa a \end{vmatrix} = 0 \Rightarrow e^{-\kappa a}\cosh \kappa a + \left(1 - \frac{\lambda}{\kappa a}\right)e^{-\kappa a}\sinh \kappa a = 0 \Rightarrow \underline{\underline{\coth \kappa a = \frac{\lambda}{\kappa a} - 1}}$$

If this condition is satisfied, A can be written in terms of B as $\underline{\underline{A = Be^{\kappa a}\sinh \kappa a}}$.

B has to be determined by normalization

(d) (10 points)
How many bound states are there for (i) small λ (ii) large λ?

Hint: You may want to re-write the eigenvalue condition given in part (c) as:

$$\tanh \kappa a = \frac{\lambda}{\kappa a} - 1 \qquad \text{for even solution}$$

$$\tanh \kappa a = \left(\frac{\lambda}{\kappa a} - 1 \right)^{-1} \qquad \text{for odd solution}$$

Let $y = \kappa a$. To answer this question we have to investigate how many solutions are there in

$$\tanh y = \frac{\lambda}{y} - 1 \qquad \text{for even wave funtion}$$

$$\tanh y = \left(\frac{\lambda}{y} - 1\right)^{-1} \qquad \text{for odd wave function}$$

Even case :

As y varies from 0 to λ, $\frac{\lambda}{y} - 1$ is monotonic deceasing from ∞ to 0,

tanh y is monotonic increasing from 0. Hence the two curves must meet once.
In other words, there is always exactly one and only one even solution for the Schoerdinger equation.

Odd case :

This is more complicated because $\left(\frac{\lambda}{y} - 1\right)^{-1}$ is also monotonic increasing. However, note the following.

1. Both tanh y and $\left(\frac{\lambda}{y} - 1\right)^{-1}$ equal to 0 at $y = 0$.

2. Second derivative of $\left(\frac{\lambda}{y} - 1\right)^{-1}$ is positive while it is negative for tanh y.

$\therefore$ They will meet (once) only if the slope of $\left(\frac{\lambda}{y} - 1\right)^{-1}$ is less than that of tanh y at $y = 0$. i.e. there is only one solution if

$$\frac{d\left(\frac{\lambda}{y} - 1\right)^{-1}}{dy} \le \frac{d(\tanh y)}{dy} \text{ at } y = 0 \quad \Rightarrow \quad \frac{\lambda}{y^2} \Big/ \left(\frac{\lambda}{y} - 1\right)^2 \le \frac{4}{\left(e^{y} + e^{-y}\right)^2} \quad \text{at } y = 0$$

$$\Rightarrow \frac{1}{\lambda} \le 1$$

$$\Rightarrow \lambda \ge 1$$

$\therefore$ There is one odd solution if $\lambda \ge 1$ (large λ), and no odd solution if $\lambda < 1$ (small λ).