Name:_______Solution________________

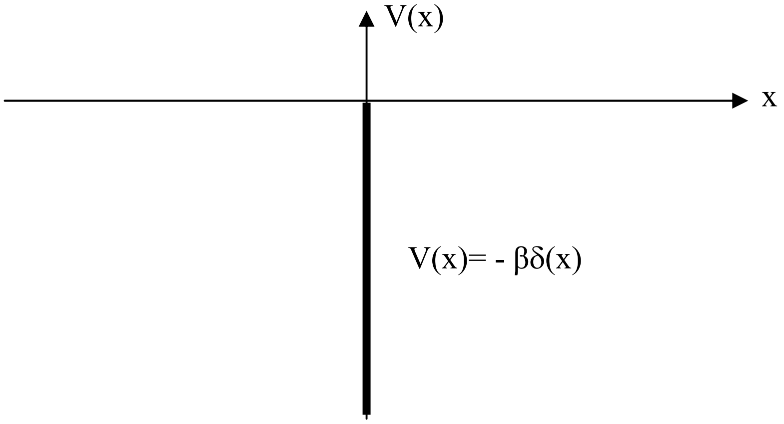
University of Kentucky
Department of Physics and Astronomy

PHY 520 Introduction to Quantum Mechanics
Fall 2004
Final Examination

Answer all four questions (total 200 points). Write down all work in detail.
Time allowed: 120 minutes
Merry Christmas and Happy New Year!

1. (50 points)

Consider a particle of mass moving under the influence of a delta function potential V(x)= -βδ(x) with energy E<0.



(a) (10 points)
Write down the boundary conditions for the eigenfunction at x=0.

Continuity of wave function $\Rightarrow \psi(-\varepsilon) = \psi(\varepsilon) \qquad (\varepsilon \to 0)$

Schroedinger equation :

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) - \phi\delta(x)\psi(x) = E\psi(x) \Rightarrow -\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{\partial}{\partial x}\psi(x)dx - \int_{-\varepsilon}^{\varepsilon}\beta\delta(x)\psi(x)dx = \int_{-\varepsilon}^{\varepsilon}E\psi(x)dx$$

$$\Rightarrow \frac{\hbar^2}{2m}\left[\psi'(\varepsilon) - \psi'(-\varepsilon)\right] + \beta\psi(0) = 0 \qquad (\varepsilon \to 0)$$

(b) (20 points)

Solve for the eigenfunction(s) and the corresponding energy eigenvalue(s).
How many bound states are there?

Since $E < 0$, let $\kappa = \sqrt{\frac{2m\,|\,E\,|}{\hbar^2}}$

The wave function for $x < 0$ is $\psi_{<} = Ae^{\kappa x}$

The wave function for $x > 0$ is $\psi_{>} = Be^{-\kappa x}$

$\psi(\text{-}\varepsilon) = \psi(\varepsilon) \Rightarrow A = B$

$\therefore$ The wave function for $x > 0$ is $\psi_{>} = Ae^{-\kappa x}$ (i.e. the wavefunction is even)

Schroedinger equation :

$$\frac{\hbar^2}{2m}\left[\psi'(\varepsilon) - \psi'(-\varepsilon)\right] + \beta\psi(0) = 0 \Rightarrow \frac{\hbar^2}{2m}\left[(-\kappa A) - (\kappa A)\right] + \beta A = 0$$

$$\Rightarrow \frac{\hbar^2 \kappa}{m} = \beta$$

$$\Rightarrow \sqrt{\frac{2m\,|\,E\,|}{\hbar^2}} = \frac{m\beta}{\hbar^2}$$

$$\Rightarrow \frac{2m\,|\,E\,|}{\hbar^2} = \frac{m^2\beta^2}{\hbar^4}$$

$$\Rightarrow E = -\frac{m\lambda^2}{2\hbar^2}$$

To determine A :

$$\int_{-\infty}^{0} \left(Ae^{\kappa x}\right)^2 dx + \int_{0}^{\infty} \left(Ae^{-\kappa x}\right)^2 dx = 1 \Rightarrow \left[\frac{A^2}{2\kappa}e^{2\kappa x}\right]_{-\infty}^{0} + \left[\frac{A^2}{2\kappa}e^{-2\kappa x}\right]_{0}^{\infty} = 1$$

$$\Rightarrow \frac{A^2}{2\kappa} + \frac{A^2}{2\kappa} = 1$$

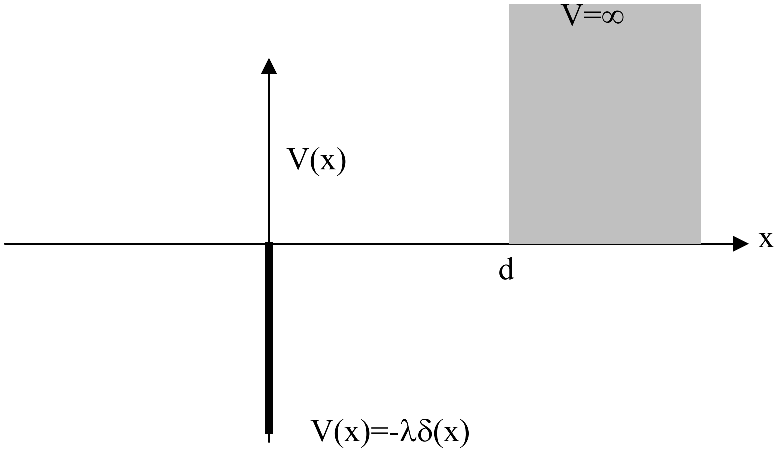
$$\Rightarrow A = \sqrt{\kappa} = \frac{\sqrt{m\beta}}{\hbar}$$

There is always one bound state for single delta function potential.

(c) (20 points)

A wall is now placed at x=d so that the potential is given by

$$V(x) = \begin{cases} \infty & x > d \\ -\lambda\delta(c) & x \le d \end{cases}$$



Solve for the eigenfunction(s) and the corresponding energy eigenvalue(s) for the bound state(s). (To save time, you do not need to normalize the state function). Can you reduce your result to that of part (b) by letting d →∞ ?

Since $E < 0$, let $\kappa = \sqrt{\dfrac{2m\,|E|}{\hbar^2}}$

The wave function for $x < 0$ is $\psi_< = Ae^{\kappa x}$

The wave function for $x > 0$ is $\psi_> = Ce^{-\kappa x} + De^{\kappa x}$ (since now x will not equal to ∞)

$\psi(d) = 0 \Rightarrow Ce^{-\kappa d} + De^{\kappa d} = 0 \Rightarrow C = -De^{2\kappa d}$ ---(1)

$\psi(-\varepsilon) = \psi(\varepsilon) \Rightarrow A = C + D \Rightarrow A = (1 - e^{2\kappa d})D$ ---(2)

Therefore the wavefunction is
$$\psi(x) = \begin{cases} 0 & x \ge d \\ D(-e^{2\kappa d}e^{-\kappa x} + e^{\kappa x}) & 0 < x \le d \\ (1 - e^{2\kappa d})De^{\kappa x} & x \le 0 \end{cases}$$

D is to be determined by normalization condition :

$$\int \psi^* \psi \, dx = 1$$

Schroedinger equation :

$$\frac{\hbar^2}{2m}\left[\psi'(\varepsilon) - \psi'(-\varepsilon)\right] + \beta\psi(0) = 0 \Rightarrow \frac{\hbar^2}{2m}\left[D(\kappa e^{2\kappa d} + \kappa) - D\kappa(1 - e^{2\kappa d})\right] + \beta(1 - e^{2\kappa d})D = 0$$

$$\Rightarrow \frac{\hbar^2}{2m}\left[2\kappa e^{2\kappa d}\right] + \beta(1 - e^{2\kappa d}) = 0$$

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$$\Rightarrow \kappa = \frac{m}{\hbar^2}\beta(e^{2\kappa d} - 1)e^{-2\kappa d}$$

$$\Rightarrow \kappa = \frac{m\beta}{\hbar^2}(1 - e^{-2\kappa d})$$

As $d \to \infty$, $e^{-2\kappa d} \to 0$ and $\kappa \to \frac{m\beta}{\hbar^2}$, which is the same result as part (b).

2. (50 points)

Consider a particle of mass m in a simple harmonic potential

$$V(x)=\frac{1}{2}kx^2$$

(a) (5 points)

Write down the ground state energy *in terms of m and k.*

$$E_0=\frac{1}{2}\hbar\omega,\text{ but }\omega=\sqrt{\frac{k}{m}},\therefore E_0=\frac{1}{2}\hbar\omega=\frac{1}{2}\hbar\sqrt{\frac{k}{m}}$$

(b) (5 points)

Which of the following functions is the ground state wave function of simple harmonic oscillation? Briefly explain (qualitatively) why you choose your answer.

A. $\frac{1}{\sqrt{2}}\sin\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\xi+\frac{1}{\sqrt{2}}\cos\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\xi$

B. $2\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\xi$

C. $2\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\xi\, e^{-\xi^2/2}$

D. $2\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\xi\, e^{-\xi/2}$

[✓ E. $\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\xi^2/2}$]

F. $\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\xi/2}$

G. $\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} P_{\ell}^{m}(\xi)$

H. $\frac{1}{\sqrt{8}}\left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}(4\xi^2-2)e^{-\xi^2/2}$

In above, $\xi=\sqrt{\frac{m\omega}{\hbar}}x$, and P_{ℓ}^{m} is the Associated Legendre Polynomials.

(c) (10 points)

Calculate <x> and <p> for the ground state, where p is momentum.

$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\xi^2/2}$$

$$\text{Let } \xi = \sqrt{\frac{m\omega}{\hbar}}\, x = \alpha x \text{ with } \alpha = \sqrt{\frac{m\omega}{\hbar}}$$

$$\psi(x) = \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} e^{-\alpha^2 x^2/2}$$

$$< x >= \int_{-\infty}^{\infty} \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} e^{-\alpha^2 x^2/2} x \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} e^{-\alpha^2 x^2/2} dx = \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \underbrace{e^{-\alpha^2 x^2} x}_{\text{Odd function}} dx = 0$$

$$< p >= \int_{-\infty}^{\infty} \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} e^{-\alpha^2 x^2/2} \left(-i\hbar \frac{\partial}{\partial x}\right) \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} e^{-\alpha^2 x^2/2} dx$$

$$= -i\hbar \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-\alpha^2 x^2/2} \left(-\alpha^2 x\, e^{-\alpha^2 x^2/2}\right) dx$$

$$= i\hbar\alpha^2 \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \underbrace{x e^{-\alpha^2 x^2}}_{\text{Odd function}} dx = 0$$

(d) (15 points)

Calculate $<x^2>$ and $<p^2>$. You may find the following integration useful:

$$\int_{-\infty}^{\infty} x^{2n} e^{-\alpha^2 x^2} dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n - 1)}{2^n \alpha^{2n+1}} \sqrt{\pi}$$

$$\psi(x) = \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} e^{-\alpha^2 x^2/2}$$

$$< x^2 >= \int_{-\infty}^{\infty} \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} e^{-\alpha^2 x^2/2} \cdot x^2 \cdot \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} e^{-\alpha^2 x^2/2} dx$$

$$= \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} x^2\, e^{-\alpha^2 x^2} dx$$

Note that $d(xe^{-\alpha^2x^2}) = e^{-\alpha^2x^2} - 2\alpha^2x^2e^{-\alpha^2x^2}$

$$\Rightarrow xe^{-\alpha^2x^2} = \int e^{-\alpha^2x^2}dx - 2\alpha^2\int x^2e^{-\alpha^2x^2}dx$$

$$\Rightarrow \int_{-\infty}^{\infty} x^2e^{-\alpha^2x^2}dx = \frac{1}{2\alpha^2}\int_{-\infty}^{\infty} e^{-\alpha^2x^2}dx = \frac{1}{2\alpha^2}\frac{\sqrt{\pi}}{\alpha} = \frac{\sqrt{\pi}}{2\alpha^3}$$

$$\therefore < x^2 >= \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{2}} \cdot \frac{\sqrt{\pi}}{2\alpha^3} = \frac{1}{2\alpha^2} = \underline{\underline{\frac{1}{2}\frac{\hbar}{m\omega}}} \qquad (\because \alpha = \sqrt{\frac{m\omega}{\hbar}})$$

$$|\psi(x)>= \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\xi^2/2}$$

Let $\xi = \sqrt{\frac{m\omega}{\hbar}}\,x = \alpha x$ with $\alpha = \sqrt{\frac{m\omega}{\hbar}}$

$$|\psi(x)>= \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} e^{-\alpha^2x^2/2}$$

$$< p^2 >=< \psi(x)|\left(-\hbar^2\frac{\partial^2}{\partial x^2}\right)|\psi(x)>$$

$$= \int_{-\infty}^{\infty}\left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} e^{-\alpha^2x^2/2} \cdot -\hbar^2\frac{\partial^2}{\partial x^2} \cdot \left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{4}} e^{-\alpha^2x^2/2}dx$$

$$= -\hbar^2\left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty} e^{-\alpha^2x^2/2}\frac{\partial}{\partial x}\left[-\alpha^2xe^{-\alpha^2x^2/2}\right]dx$$

$$= \hbar^2\alpha^2\left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty} e^{-\alpha^2x^2/2}\frac{\partial}{\partial x}\left[xe^{-\alpha^2x^2/2}\right]dx$$

$$= \hbar^2\alpha^2\left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{2}}\int_{-\infty}^{\infty} e^{-\alpha^2x^2/2}\left[e^{-\alpha^2x^2/2} - \alpha^2x^2e^{-\alpha^2x^2/2}\right]dx$$

$$= \hbar^2\alpha^2\left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{2}}\left[\int_{-\infty}^{\infty} e^{-\alpha^2x^2}dx - \alpha^2\int_{-\infty}^{\infty} x^2e^{-\alpha^2x^2}dx\right]$$

$$\int_{-\infty}^{\infty} e^{-\alpha^2x^2}dx = \left[\int_{-\infty}^{\infty} e^{-\alpha^2x^2}dx\int_{-\infty}^{\infty} e^{-\alpha^2y^2}dy\right]^{1/2} = \left[\int_{0}^{\infty} e^{-\alpha^2r^2}rdr\int_{0}^{2\pi} d\theta\right]^{1/2}$$

$$= \left[2\pi\int_{0}^{\infty} e^{-\alpha^2u}\frac{1}{2}du\right]^{1/2} = \left\{\pi\left[\frac{e^{-\alpha^2u}}{-\alpha^2}\right]_{0}^{\infty}\right\}^{1/2} = \frac{\sqrt{\pi}}{\alpha}$$

On the other hand, $\mathrm{d}(\mathrm{x}e^{-\alpha^2 x^2}) = e^{-\alpha^2 x^2} - 2\alpha^2 x^2 e^{-\alpha^2 x^2}$

$$\Rightarrow \mathrm{x}e^{-\alpha^2 x^2} = \int e^{-\alpha^2 x^2}\mathrm{dx} - 2\alpha^2 \int x^2 e^{-\alpha^2 x^2}\mathrm{dx}$$

$$\Rightarrow \int_{-\infty}^{\infty} x^2 e^{-\alpha^2 x^2}\mathrm{dx} = \frac{1}{2\alpha^2}\int_{-\infty}^{\infty} e^{-\alpha^2 x^2}\mathrm{dx} = \frac{1}{2\alpha^2}\frac{\sqrt{\pi}}{\alpha} = \frac{\sqrt{\pi}}{2\alpha^3}$$

Substitute these integrations into the last equation :

$$< p^2 >= \hbar^2\alpha^2\left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{2}}\left[\frac{\sqrt{\pi}}{\alpha} - \alpha^2\frac{\sqrt{\pi}}{2\alpha^3}\right]$$

$$\therefore < p^2 >= \hbar^2\alpha^2\left(\frac{\alpha^2}{\pi}\right)^{\frac{1}{2}}\left[\frac{\sqrt{\pi}}{2\alpha}\right] = \frac{1}{2}\hbar^2\alpha^2 = \frac{1}{2}\hbar^2\frac{\mathrm{m}\omega}{\hbar} \qquad (\because \alpha = \sqrt{\frac{\mathrm{m}\omega}{\hbar}})$$

$$= \underline{\underline{\frac{1}{2}\hbar \mathrm{m}\omega}}$$

(e) (15 points)
Calculate $\sigma_x\sigma_p$ to show that Uncertainty Principle is followed. Also show that <T> = <V>, where T is kinetic energy.

$$\sigma_x\sigma_p = \sqrt{< x^2 > - < x >^2}\sqrt{< p^2 > - < p >^2} = \sqrt{\frac{1}{2}\frac{\hbar}{\mathrm{m}\omega} - 0^2}\sqrt{\frac{1}{2}\hbar \mathrm{m}\omega - 0^2}$$

$$= \sqrt{\left(\frac{1}{2}\frac{\hbar}{\mathrm{m}\omega}\right)\left(\frac{1}{2}\hbar \mathrm{m}\omega\right)}$$

$$= \underline{\underline{\frac{\hbar}{2}}}$$

This satisfies the Uncertainty Principle $\sigma_x\sigma_p \geq \frac{\hbar}{2}$

$$< T >= \frac{< p^2 >}{2\mathrm{m}} = \frac{1}{2\mathrm{m}}\left(\frac{1}{2}\hbar \mathrm{m}\omega\right) = \frac{1}{4}\hbar\omega$$

$$< V >= \frac{1}{2}\mathrm{k} < x^2 >= \frac{1}{2}\mathrm{k}\left(\frac{1}{2}\frac{\hbar}{\mathrm{m}\omega}\right) = \mathrm{k}\frac{\hbar}{4\mathrm{m}\omega}$$

But $\omega^2 = \frac{\mathrm{k}}{\mathrm{m}} \Rightarrow \mathrm{k} = \mathrm{m}\omega^2$

$$\therefore < V >= \mathrm{m}\omega^2 \cdot \frac{\hbar}{4\mathrm{m}\omega} = \frac{1}{4}\hbar\omega =< T >$$

3. (50 points)

An electron is at rest in an oscillating magnetic field

$$\vec{B} = B_0 \cos(\omega t)\,\hat{k} \qquad (\hat{k} \text{ is the unit vector along the z - axis})$$

Where B_0 and ω are constants.

(a) (10points)
Construct the Hamiltonian matrix for this system.

$$H = -\vec{\mu}\cdot\vec{B} = -\gamma\vec{s}\cdot\vec{B} = -\gamma\,\hat{s}_z B_z = -\frac{\hbar}{2}\gamma B_0 \cos(\omega t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(b) (20 points)
The electron starts out (at t=0) in the spin-up state wotj res[ect tp the x-axis (that is: $\chi(0) = \chi_+^{(x)}$). Determine $\chi(t)$ at any subsequent time from the time dependent Schroedinger equation.

We will work with S_z – representation in solving this problem.

Since the electron starts out at spin up state with respect to the x - axis, we need to find out the eigenstate of $\hat{S}_x$ correspond to an eigenvalue of $\frac{\hbar}{2}$ (i.e. "spinu up") first :

$$\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = +\frac{\hbar}{2}\begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \begin{cases} v = u \\ u = v \end{cases}$$

If we choose u = 1, then v = 1, i.e. the eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Normalizing this vector, we obtain the normalized eigenvector as $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

$$\therefore \chi(0) = \chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (\text{in } S_z - \text{representation})$$

To determine $\chi(0)$, we have to solve the time dependent Schroedinger equation :

$$i\hbar\frac{\partial}{\partial t}\chi(t) = \hat{H}\,\chi(t) \Rightarrow i\hbar\frac{\partial}{\partial t}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = -\frac{\hbar}{2}\gamma B_0 \cos(\omega t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$$

$$\Rightarrow \frac{\partial}{\partial t}\begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{i}{2}\gamma B_0 \cos(\omega t)\begin{pmatrix} a(t) \\ -\,b(t) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \begin{pmatrix} a(0)e^{\frac{i}{2\omega}\gamma B_0 \sin(\omega t)} \\ b(0)e^{-\frac{i}{2\omega}\gamma B_0 \sin(\omega t)} \end{pmatrix}$$

But we know that $\begin{pmatrix} a(0) \\ b(0) \end{pmatrix} = \chi(0) = \chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ $, \therefore a(0) = \frac{1}{\sqrt{2}}, b(0) = \frac{1}{\sqrt{2}}.$

$$\therefore \chi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{\frac{i}{2\omega}\gamma B_0 \sin(\omega t)} \\ e^{-\frac{i}{2\omega}\gamma B_0 \sin(\omega t)} \end{pmatrix} \text{ or } \frac{1}{\sqrt{2}} e^{\frac{i}{2\omega}\gamma B_0 \sin(\omega t)} |\uparrow> + \frac{1}{\sqrt{2}} e^{-\frac{i}{2\omega}\gamma B_0 \sin(\omega t)} |\downarrow>$$

(c) (15 points)
Find the probability of getting $+\hbar/2$, if you measure S_x.

We have alredy determined in part (b) that the eigenvector for S_x corresponds to an eigenvalue of $\frac{\hbar}{2}$ is :

$$\chi_+^{(x)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad (\text{in } S_z - \text{representation})$$

$\therefore$ Probability of getting $\frac{\hbar}{2}$ if you measure S_z is :

$$\left| < \chi_+^{(x)} \mid \chi(t) > \right|^2 = \left| \frac{1}{\sqrt{2}}(1 \quad 1) \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{i}{2\omega}\gamma B_0 \sin(\omega t)} \\ e^{-\frac{i}{2\omega}\gamma B_0 \sin(\omega t)} \end{pmatrix} \right|^2$$

$$= \left| \frac{e^{\frac{i}{2\omega}\gamma B_0 \sin(\omega t)} + e^{-\frac{i}{2\omega}\gamma B_0 \sin(\omega t)}}{2} \right|^2$$

$$= \left| \cos\left(\frac{1}{2\omega}\gamma B_0 \sin(\omega t) \right) \right|^2$$

$$= \cos^2\left(\frac{\gamma B_0}{2\omega} \sin(\omega t) \right)$$

(d) (5 points)
What is the minimum field (B_0) required to force a complete flip in S_x?

From part (c) above, we can see the system is oscillating between $\chi_+^{(x)}$ and $\chi_-^{(x)}$. For a complete flip in S_x, the above probability should be able to reach 0 at a certain time. This requires

$$\frac{\gamma B_0}{2\omega} \geq \frac{\pi}{2} \ \Rightarrow \ B_0 \geq \frac{\omega \pi}{\gamma}$$

4. (50 points)

In doing this problem, you may find the following Clebsch-Gordon coefficient table for adding $|j_1,m_1>$ and $|j_2=1/2,m_2>$ into $|j, m_J>$:

	$m_1, m_2 = \frac{1}{2}$	$m_1, m_2 = -\frac{1}{2}$
$j = j_1 + \frac{1}{2}, m_J = m_1 + m_2$	$\sqrt{\frac{j_1 + m + \frac{1}{2}}{2j_1 + 1}}$	$\sqrt{\frac{j_1 - m + \frac{1}{2}}{2j_1 + 1}}$
$j = j_1 - \frac{1}{2}, m_J = m_1 + m_2$	$-\sqrt{\frac{j_1 - m + \frac{1}{2}}{2j_1 + 1}}$	$\sqrt{\frac{j_1 + m + \frac{1}{2}}{2j_1 + 1}}$

The state function of an electron in a hydrogen atom is described as

$$\Psi = \sqrt{\frac{2}{5}}\psi_{510} + \sqrt{\frac{3}{5}}\psi_{511}$$

and it is known that the electron is in a spin up state (i.e. m=1/2).

(a) (10 points)

What are the possible values of total angular momentum quantum number j and its z-component m_J?

It is known that j_1=1and j_2=1/2, hence j =1 -1/2=1/2 or j=1+1/2=3/2 (i.e. two possible values for j)

In general, if j=1/2, m_J=-1/2 or +1/2. If j=3/2. , m_J=-3/2 or -1/2, 1/2, 3/2. Hence there are 4 possible values of m_J (-3/2 or -1/2, 1/2, 3/2). However, it is necessary to satisfy $m_J = m_1 + m_2$. m_J for ψ_{510} can only be 0+1/2=+1/2, and m_J for ψ_{511} can only be 1+1/2=3/2. There are only two possible values of m_J for the present problem.+1/2 and +3/2.

In other words, we can expressed Ψ in the following three total angular momentum states:

$|j=1/2, m_J=+1/2>$ (note that $|j=1/2, m_J=+3/2>$ is not possible)
$|j=3/2, m_J=+1/2>$ and
$|j=3/2, m_J=+3/2>$

(b) (10 points)

Make use of the above given Clebsch-Gordon coefficients, construct *two* tables for the cases (i) m_1=0, m_2=1/2, and (ii) m_1=1, m_2=1/2.

(i) For the case m_l=0, m_2=1/2:

	$m_1 = 0, m_2 = \frac{1}{2}$ $(m_J = \frac{1}{2})$	$m_1 = 0, m_2 = -\frac{1}{2}$ $(m_J = -\frac{1}{2})$
$j = \frac{3}{2}, m_J = m_1 + m_2$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$
$j = \frac{1}{2}, m_J = m_1 + m_2$	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$

(ii) For the case m_l=1, m_2=1/2.

	$m_1 = 1, m_2 = \frac{1}{2}$ $(m_J = \frac{3}{2})$	$m_1 = 1, m_2 = -\frac{1}{2}$ $(m_J = \frac{1}{2})$
$j = \frac{3}{2}, m_J = m_1 + m_2$	1	$\sqrt{\frac{1}{3}}$
$j = \frac{1}{2}, m_J = m_1 + m_2$	0	$\sqrt{\frac{2}{3}}$

Note that we have "overdone" the calculation. The last column in above tables correspond to the case m_2= -1/2, which is not necessary for this problem (since the electron is in the spin up state).

(c) (10 points)
Make use of your results in part (b), write Ψ in terms of the total angular momentum eigenvectors |j, m_J>.

Using the table for m$_l$=0

`	$m_1 = 0, m_2 = \frac{1}{2}$ $(m_J = \frac{1}{2})$	$m_1 = 0, m_2 = -\frac{1}{2}$ $(m_J = -\frac{1}{2})$
$j = \frac{3}{2}, m_J = m_1 + m_2$	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$
$j = \frac{1}{2}, m_J = m_1 + m_2$	$-\sqrt{\frac{1}{3}}$	$\sqrt{\frac{1}{3}}$

$$\therefore\ \psi_{510} = \sqrt{\frac{2}{3}}\,|\, j = \tfrac{3}{2}, m_J = \tfrac{1}{2} > -\sqrt{\frac{1}{3}}\,|\, j = \tfrac{1}{2}, m_J = \tfrac{1}{2} >$$

Using the table for m$_l$=1:

	$m_1 = 1, m_2 = \frac{1}{2}$ $(m_J = \frac{3}{2})$	$m_1 = 1, m_2 = -\frac{1}{2}$ $(m_J = \frac{1}{2})$
$j = \frac{3}{2}, m_J = m_1 + m_2$	1	$\sqrt{\frac{1}{3}}$
$j = \frac{1}{2}, m_J = m_1 + m_2$	0	$\sqrt{\frac{2}{3}}$

$$\therefore\ \psi_{511} = |\, j = \tfrac{3}{2}, m_J = \tfrac{3}{2} >$$

$$\Psi = \sqrt{\frac{2}{5}}\psi_{510} + \sqrt{\frac{3}{5}}\psi_{511}$$

$$= \sqrt{\frac{2}{5}}\left(\sqrt{\frac{2}{3}}\,|\, j = \tfrac{3}{2}, m_J = \tfrac{1}{2} > -\sqrt{\frac{1}{3}}\,|\, j = \tfrac{1}{2}, m_J = \tfrac{1}{2} > \right) + \sqrt{\frac{3}{5}}\,|\, j = \tfrac{3}{2}, m_J = \tfrac{3}{2} >$$

$$= \sqrt{\frac{4}{15}}\,|\, j = \tfrac{3}{2}, m_J = \tfrac{1}{2} > -\sqrt{\frac{2}{15}}\,|\, j = \tfrac{1}{2}, m_J = \tfrac{1}{2} > +\sqrt{\frac{3}{5}}\,|\, j = \tfrac{3}{2}, m_J = \tfrac{3}{2} >$$

(d) (10 points)
If the total angular momentum is measured, what are the probabilities of obtaining (i) j=1/2 and (ii) j=3/2?

$$\Psi = \sqrt{\frac{4}{15}}\,|\,j=\tfrac{3}{2}, m_J = \tfrac{1}{2}> - \sqrt{\frac{2}{15}}\,|\,j=\tfrac{1}{2}, m_J = \tfrac{1}{2}> + \sqrt{\frac{3}{5}}\,|\,j=\tfrac{3}{2}, m_J = \tfrac{3}{2}>$$

$$\therefore \text{ Probability } (j = \frac{1}{2}) = \left(\sqrt{\frac{2}{15}}\right)^2 = \frac{2}{15}$$

$$\text{Probability } (j = \frac{3}{2}) = \left(\sqrt{\frac{4}{15}}\right)^2 + \left(\sqrt{\frac{3}{5}}\right)^2 = \frac{13}{15}$$

(e) (10 points)
If the z-component of the total angular momentum is measured, what are the probabilities of obtaining (i) m_J=-3/2, (ii) m_J=-1/2, (iii) m_J=1/2, and m_J=3/2 ?

$$\Psi = \sqrt{\frac{4}{15}}\,|\,j=\tfrac{3}{2}, m_J = \tfrac{1}{2}> - \sqrt{\frac{2}{15}}\,|\,j=\tfrac{1}{2}, m_J = \tfrac{1}{2}> + \sqrt{\frac{3}{5}}\,|\,j=\tfrac{3}{2}, m_J = \tfrac{3}{2}>$$

$$\therefore \text{ Probability } (m_j = -\frac{3}{2}) = 0$$

$$\text{Probability } (m_j = -\frac{1}{2}) = 0$$

$$\text{Probability } (m_j = \frac{1}{2}) = \left(\sqrt{\frac{4}{15}}\right)^2 + \left(\sqrt{\frac{2}{15}}\right)^2 = \frac{2}{3}$$

$$\text{Probability } (m_j = \frac{3}{2}) = \left(\sqrt{\frac{3}{5}}\right)^2 = \frac{3}{5}$$