University of Kentucky
Department of Physics and Astronomy

PHY 520 Introduction to Quantum Mechanics
Fall 2004
Test 1

Name: ____________KEY________________________

Answer all questions. Write down all work in detail.
Time allowed: 50 minutes

1. (50 points)

A particle of mass m is confined to a one dimensional region –a ≤ x ≤ a with an infinite potential

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \le x \le a \\ \infty & x > a \end{cases}$$

The wave function at t=0 is given as:

$$\Psi(x,t=0) = A\left[\psi_1(x) + 2\ \psi_2(x)\right]$$

where $\psi_1(x)$ is the ground state and $\psi_2(x)$ is the first excited state.

Answer all of the following.

(a) (10 points)
Write down all the normalized energy eigenstates $\psi_n(x)$ and the corresponding energy E_n.

$$\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(k_n x\right) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right)$$

$$E_n = \frac{\hbar^2 k_n{}^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2ma^2}$$

(b) (15 points)

Determine the value of A so that the state function $\Psi(x,0)$ is normalized.

$$<\Psi(x,t=0)\,|\,\Psi(x,t=0)>=1 \Rightarrow \mathrm{A}^2 <\psi_1(x)+2\ \psi_2(x)\,|\,\psi_1(x)+2\ \psi_2(x)>=1$$

$$\Rightarrow \mathrm{A}^2 [<\psi_1(x)\,|\,\psi_1(x)>+4<\psi_2(x)\,|\,\psi_2(x)>]=1$$

$$\Rightarrow 5\mathrm{A}^2=1$$

$$\Rightarrow \mathrm{A}=\sqrt{\frac{1}{5}}$$

$$\therefore \Psi(x,t=0)=\sqrt{\frac{1}{5}}\,\psi_1(x)+2\sqrt{\frac{1}{5}}\psi_2(x)$$

(c) (15 points)

What is <E> at t=0?

$$\Psi(x,t=0)=\sqrt{\frac{1}{5}}\,\psi_1(x)+2\sqrt{\frac{1}{5}}\psi_2(x)$$

$$\therefore <\mathrm{E}>=\left(\sqrt{\frac{1}{5}}\right)^2 \mathrm{E}_1+\left(2\sqrt{\frac{1}{5}}\right)^2 \mathrm{E}_2$$

$$=\frac{1}{5}\frac{\hbar^2 1^2 \pi^2}{2\mathrm{ma}^2}+\frac{4}{5}\frac{\hbar^2 2^2 \pi^2}{2\mathrm{ma}^2}$$

$$=\frac{17}{10}\frac{\hbar^2 \pi^2}{\mathrm{ma}^2}$$

(d) (10 points)

What is <E> at any other time t?

<E> should be a constant of time, i.e.

$$<\mathrm{E}>=\frac{17}{10}\frac{\hbar^2\pi^2}{\mathrm{ma}^2} \text{ at any time t.}$$

2. (50 points)

The first three energy eigenstates of a simple harmonic potential $V(x)=m\omega^2x^2/2$ are given as:

$$\psi_0 = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}(1)e^{-\xi^2/2}$$

$$\psi_1 = \sqrt{2}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}\xi e^{-\xi^2/2}$$

$$\psi_2 = \frac{1}{\sqrt{2}}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}\left(2\xi^2-1\right)e^{-\xi^2/2}$$

where

$$\xi = \sqrt{\frac{m\omega}{\hbar}}\,x$$

You may find the following intergration useful for this problem :

$$\int_0^\infty e^{-\alpha u^2}du = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

$$\int_0^\infty ue^{-\alpha u^2}du = \frac{1}{2\alpha}$$

$$\int_0^\infty u^n e^{-\alpha u^2}du = \frac{n-1}{2\alpha}\int_0^\infty u^{n-2}e^{-\alpha u^2}du$$

The wavefunction os a particle at t=0 is given as:

$$\Psi(x,0) = \frac{1}{3}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}\left(4\xi^2-1\right)e^{-\xi^2/2}$$

(a) (20 points)
Write $\Psi(x,0)$ as a linear combination of ψ_0, ψ_1, and ψ_2.

$|\Psi(x,0)\rangle = \sum_n |\psi_n(x)\rangle\langle\psi_n(x)|\Psi(x,0)\rangle$ because $\sum_n |\psi_n(x)\rangle\langle\psi_n(x)| = 1$

$$\langle\psi_0(x)|\Psi(x,0)\rangle = \int_{-\infty}^{\infty}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}e^{-\xi^2/2}\,\frac{1}{3}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}\left(4\xi^2-1\right)e^{-\xi^2/2}\,dx$$

$$< \psi_0(x) | \Psi(x,0) > = \frac{1}{3}\left(\frac{\mathrm{m}\omega}{\hbar\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left(4\xi^2 - 1\right) e^{-\xi^2} \,\mathrm{dx}$$

$$= \frac{1}{3}\left(\frac{\mathrm{m}\omega}{\hbar\pi}\right)^{\frac{1}{2}} \sqrt{\frac{\hbar}{\mathrm{m}\omega}} \int_{-\infty}^{\infty} \left(4\xi^2 - 1\right) e^{-\xi^2} \,\mathrm{d}\xi$$

$$= \frac{1}{3\sqrt{\pi}}\left[\int_{-\infty}^{\infty} 4\xi^2 e^{-\xi^2} \,\mathrm{d}\xi - \int_{-\infty}^{\infty} e^{-\xi^2} \,\mathrm{d}\xi\right]$$

$$= \frac{2}{3\sqrt{\pi}}\left[\int_{0}^{\infty} 4\xi^2 e^{-\xi^2} \,\mathrm{d}\xi - \int_{0}^{\infty} e^{-\xi^2} \,\mathrm{d}\xi\right]$$

$$= \frac{2}{3\sqrt{\pi}}\left[4 \cdot \frac{2-1}{2} - 1\right] \int_{-\infty}^{\infty} e^{-\xi^2} \,\mathrm{d}\xi$$

$$= \frac{2}{3\sqrt{\pi}}\left[4 \cdot \frac{2-1}{2} - 1\right] \frac{\sqrt{\pi}}{2}$$

$$= \frac{1}{3}$$

$$< \psi_1(x) | \Psi(x,0) > = \int_{-\infty}^{\infty} \sqrt{2}\left(\frac{\mathrm{m}\omega}{\hbar\pi}\right)^{\frac{1}{4}} \xi e^{-\xi^2/2} \, \frac{1}{3}\left(\frac{\mathrm{m}\omega}{\hbar\pi}\right)^{\frac{1}{4}} \left(4\xi^2 - 1\right) e^{-\xi^2/2} \,\mathrm{dx}$$

$$= \frac{\sqrt{2}}{3}\left(\frac{\mathrm{m}\omega}{\hbar\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} \left(4\xi^3 - \xi\right) e^{-\xi^2} \,\mathrm{dx}$$

$$= 0 \qquad (\because \text{Integration of odd function from } -\infty \text{ to } \infty)$$

$$< \psi_2(x) | \Psi(x,0) > = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}\left(\frac{\mathrm{m}\omega}{\hbar\pi}\right)^{\frac{1}{4}} \left(2\xi^2 - 1\right) e^{-\xi^2/2} \, \frac{1}{3}\left(\frac{\mathrm{m}\omega}{\hbar\pi}\right)^{\frac{1}{4}} \left(4\xi^2 - 1\right) e^{-\xi^2/2} \,\mathrm{dx}$$

$$= \frac{1}{3\sqrt{2}}\left(\frac{\mathrm{m}\omega}{\hbar\pi}\right)^{\frac{1}{2}} \sqrt{\frac{\hbar}{\mathrm{m}\omega}} \int_{-\infty}^{\infty} \left(2\xi^2 - 1\right)\left(4\xi^2 - 1\right) e^{-\xi^2} \,\mathrm{d}\xi$$

$$= \frac{1}{3\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(8\xi^4 - 6\xi^2 + 1\right) e^{-\xi^2} \,\mathrm{d}\xi$$

$$= \frac{2}{3\sqrt{2\pi}}\left[\int_{0}^{\infty} 8\xi^4 e^{-\xi^2} \,\mathrm{d}\xi - \int_{0}^{\infty} 6\xi^2 e^{-\xi^2} \,\mathrm{d}\xi + \int_{0}^{\infty} e^{-\xi^2} \,\mathrm{d}\xi\right]$$

$$= \frac{2}{3\sqrt{2\pi}}\left[8 \cdot \frac{4-1}{2} \cdot \frac{2-1}{2} \int_{0}^{\infty} e^{-\xi^2} \,\mathrm{d}\xi - 6 \cdot \frac{2-1}{2} \int_{0}^{\infty} e^{-\xi^2} \,\mathrm{d}\xi + \int_{0}^{\infty} e^{-\xi^2} \,\mathrm{d}\xi\right]$$

$$= \frac{2}{3\sqrt{2\pi}}\left[8 \cdot \frac{4-1}{2} \cdot \frac{2-1}{2} - 6 \cdot \frac{2-1}{2} + 1\right] \int_{0}^{\infty} e^{-\xi^2} \,\mathrm{d}\xi$$

$$= \frac{2}{3\sqrt{2\pi}}\left[8 \cdot \frac{4-1}{2} \cdot \frac{2-1}{2} - 6 \cdot \frac{2-1}{2} + 1\right] \frac{\sqrt{\pi}}{2}$$

$$= \frac{2}{3\sqrt{2\pi}}[6-3+1]\frac{\sqrt{\pi}}{2}$$

$$= \frac{4}{3\sqrt{2}}$$

$$= \frac{2\sqrt{2}}{3}$$

$$\therefore\ \Psi(x,0) = \frac{1}{3}\psi_0(x) + \frac{2\sqrt{2}}{3}\psi_2(x)$$

(b) (15 points)
What is the wavefunction at time t (i.e. $\Psi(x,t)$) ?

$$\Psi(x,0) = \frac{1}{3}\psi_0(x) + \frac{2\sqrt{2}}{3}\psi_2(x)$$

$$\therefore\ \Psi(x,t) = \frac{1}{3}\psi_0(x)e^{-i\frac{E_0}{\hbar}t} + \frac{2\sqrt{2}}{3}\psi_2(x)e^{-i\frac{E_0}{\hbar}t}$$

$$= \frac{1}{3}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}(1)e^{-\xi^2/2}e^{-i\frac{1}{2}\omega t} + \frac{2\sqrt{2}}{3}\frac{1}{\sqrt{2}}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}(2\xi^2-1)e^{-\xi^2/2}e^{-i\frac{5}{2}\omega t}$$

$$= \frac{1}{3}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}e^{-\xi^2/2}\left[e^{-i\frac{1}{2}\omega t} + 2(2\xi^2-1)e^{-i\frac{5}{2}\omega t}\right]$$

$$= \frac{1}{3}\left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}e^{-\xi^2/2}\left[e^{-i\frac{1}{2}\omega t} - 2e^{-i\frac{5}{2}\omega t} + 4\xi^2 e^{-i\frac{5}{2}\omega t}\right]$$

(c) (15 points)
Calculate <E> at t=0 and at any other time t.

Since $\Psi(x,0) = \frac{1}{3}\psi_0(x) + \frac{2\sqrt{2}}{3}\psi_2(x)$

$$\therefore\ <E> = \left(\frac{1}{3}\right)^2 E_0 + \left(\frac{2\sqrt{2}}{3}\right)^2 E_1 = \frac{1}{9}\cdot\frac{1}{2}\hbar\omega + \frac{8}{9}\cdot\frac{5}{2}\hbar\omega = \frac{41}{18}\hbar\omega$$

<E> should be a constant of time.