University of Kentucky
Department of Physics and Astronomy

PHY 520 Introduction to Quantum Mechanics
Fall 2004
Test 2

Name: ___________________________________

Answer all questions. Write down all work in detail.
Time allowed: 50 minutes

1. (50 points)

A Consider two operators A and B in matrix form:

$$A = \begin{pmatrix} 2+2i & -2+2i \\ -2+2i & 2+2i \end{pmatrix} \qquad B = \begin{pmatrix} -2 & -6 \\ -6 & -2 \end{pmatrix}$$

(a) (10 points)
Are A and B Hermitian? Which one (can be both or none) is not physical observable?

$$A = \begin{pmatrix} 2+2i & -2+2i \\ -2+2i & 2+2i \end{pmatrix} \Rightarrow A^+ = \begin{pmatrix} 2-2i & -2-2i \\ -2-2i & 2-2i \end{pmatrix} \neq A$$

A is NOT Hermitian

$$B = \begin{pmatrix} -2 & -6 \\ -6 & -2 \end{pmatrix} \Rightarrow B^+ = \begin{pmatrix} -2 & -6 \\ -6 & -2 \end{pmatrix} = B$$

B is Hermitian

(b) (10 points)
Do A and B commute? Do they share the same set of eigenfunctions?

$$AB = \begin{pmatrix} 2+2i & -2+2i \\ -2+2i & 2+2i \end{pmatrix}\begin{pmatrix} -2 & -6 \\ -6 & -2 \end{pmatrix} = \begin{pmatrix} -4-4i+12-12i & -12-12i+4-4i \\ +4-4i-12-12i & 12-12i-4-4i \end{pmatrix} = \begin{pmatrix} 8-16i & -8-16i \\ -8-16i & 8-16i \end{pmatrix}$$

$$BA = \begin{pmatrix} -2 & -6 \\ -6 & -2 \end{pmatrix}\begin{pmatrix} 2+2i & -2+2i \\ -2+2i & 2+2i \end{pmatrix} = \begin{pmatrix} -4-4i+12-12i & +4-4i-12-12i \\ -12-12i+4-4i & +12-12i-4-4i \end{pmatrix} = \begin{pmatrix} 8-16i & -8-16i \\ -8-16i & 8-16i \end{pmatrix}$$

$\therefore$ AB = BA, i.e. A and B commute with [A.B] = 0.

Since [A,B]=0, A and B should share the same eigenvector set.

(c) (10 points)
What are the eigenvalues of A and B?

To determine eigenvalues of A:

$$\det|A - I\lambda| = 0 \Rightarrow \det\begin{pmatrix} 2+2i-\lambda & -2+2i \\ -2+2i & 2+2i-\lambda \end{pmatrix} = 0$$

$$\Rightarrow (2+2i-\lambda)^2 - (-2+2i)^2 = 0$$

$$\Rightarrow (2+2i-\lambda) = \pm(-2+2i)$$

$$\Rightarrow \lambda = 4 \text{ or } 4i \qquad \text{(So A is not Hermitian)}$$

To determine eigenvalues of B:

$$\det|B - I\lambda| = 0 \Rightarrow \det\begin{pmatrix} -2-\lambda & -6 \\ -6 & -2-\lambda \end{pmatrix}$$

$$\Rightarrow (-2-\lambda)^2 - 36 = 0$$

$$\Rightarrow (-2-\lambda) = \pm 6$$

$$\Rightarrow \lambda = -8 \text{ or } 4 \qquad \text{(So B is Hermitian)}$$

(d) (10 points)
Find the *unitary* matrix U_A and U_B so that $U_A^{-1} A\, U_A$ and $U_B^{-1} B\, U_B$ are diagonal matrices.

Since [A,B]=0 and A and B share the same eigenvector set, hence $U_A=U_B$. We just need to work on either A or B. In here we choose to find eivenvectors for B:

$$\begin{pmatrix} -2 & -6 \\ -6 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = -8\begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \begin{cases} -2u - 6v = -8u \\ -6u - 2v = -8v \end{cases} \Rightarrow \begin{cases} 6u - 6v = 0 \\ -6u + 6v = 0 \end{cases} \Rightarrow u = -v$$

Therefore one of the normalized eigen vector for B is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

We can check that this is also an eigenvector for A:

$$\begin{pmatrix} 2+2i & -2+2i \\ -2+2i & 2+2i \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. = \frac{1}{\sqrt{2}}\begin{pmatrix} 2+2i+2-2i \\ -2+2i-2-2i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 4 \\ -4 \end{pmatrix} = \frac{-4}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

To determine the other eigenvector for B :

$$\begin{pmatrix} -2 & -6 \\ -6 & -2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 4\begin{pmatrix} u \\ v \end{pmatrix} \Rightarrow \begin{cases} -2u - 6v = 4u \\ -6u - 2v = 4v \end{cases} \Rightarrow \begin{cases} -6u - 6v = 0 \\ -6u - 6v = 0 \end{cases} \Rightarrow u = v$$

Therefore one of the normalized eigen vector for B is $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

We can check that this is also an eigenvector for A :

$$\begin{pmatrix} 2+2i & -2+2i \\ -2+2i & 2+2i \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}. = \frac{1}{\sqrt{2}}\begin{pmatrix} 2+2i-2+2i \\ -2+2i+2+2i \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 4i \\ 4i \end{pmatrix} = \frac{4i}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

From these two eigengvectors, we can construct

$$U_A = U_B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

(e) (10 points)
Show explicitly that U_A and U_B are unitary. What is the reason that they are unitary?

$$U_A = U_B = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \Rightarrow U_A{}^{+} = U_B{}^{+} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

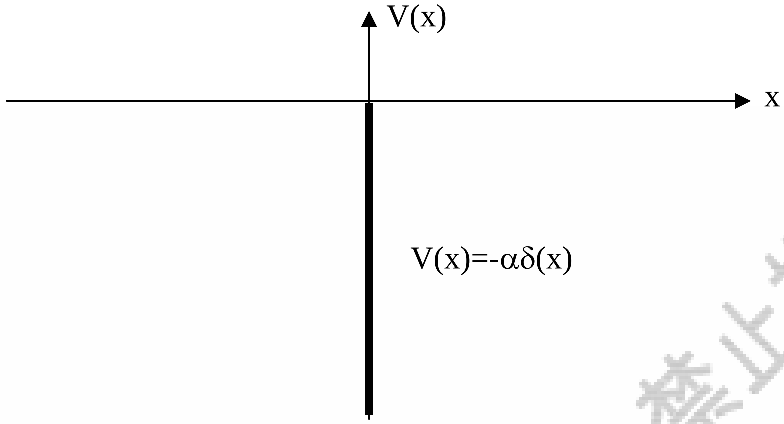
$$\therefore UU^{+} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$U^{+}U = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$\therefore$ U is unitary with $UU^{+} = U^{+}U = I$.

2. (50 points)

Consider a particle of mass moving under the influence of a delta function potential V(x)= -λδ(x) with energy E<0.



(a) (10 points)
Write down the boundary conditions for the eigenfunction at x=0.

Continuity of wave function $\Rightarrow \psi(-\varepsilon) = \psi(\varepsilon) \qquad (\varepsilon \to 0)$

Schroedinger equation :

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\psi(x) - \lambda\delta(x)\psi(x) = E\psi(x) \Rightarrow -\frac{\hbar^2}{2m}\int_{-\varepsilon}^{\varepsilon}\frac{\partial}{\partial x}\psi(x)dx - \int_{-\varepsilon}^{\varepsilon}\lambda\delta(x)\psi(x)dx = \int_{-\varepsilon}^{\varepsilon}E\psi(x)dx$$

$$\Rightarrow \frac{\hbar^2}{2m}\left[\psi'(\varepsilon) - \psi'(-\varepsilon)\right] + \lambda\psi(0) = 0 \qquad (\varepsilon \to 0)$$

(b) (20 points)
Solve for the eigenfunction(s) and the corresponding energy eigenvalue(s).
How many bound states are there?

Since $E < 0$, let $\kappa = \sqrt{\frac{2m|E|}{\hbar^2}}$

The wave function for $x < 0$ is $\psi_< = Ae^{\kappa x}$

The wave function for $x > 0$ is $\psi_> = Be^{-\kappa x}$

$\psi(-\varepsilon) = \psi(\varepsilon) \Rightarrow A = B$

$\therefore$ The wave function for $x > 0$ is $\psi_> = Ae^{-\kappa x}$ (i.e. the wavefunction is even)

Schroedinger equation :

$$\frac{\hbar^2}{2m}\left[\psi'(\varepsilon) - \psi'(-\varepsilon)\right] + \lambda\psi(0) = 0 \Rightarrow \frac{\hbar^2}{2m}\left[(-\kappa A) - (\kappa A)\right] + \lambda A = 0$$

$$\Rightarrow \frac{\hbar^2\kappa}{m} = \lambda$$

$$\Rightarrow \sqrt{\frac{2m|E|}{\hbar^2}} = \frac{m\lambda}{\hbar^2}$$

$$\Rightarrow \frac{2m|E|}{\hbar^2} = \frac{m^2\lambda^2}{\hbar^4}$$

$$\Rightarrow E = -\frac{m\lambda^2}{2\hbar^2}$$

To determine A :

$$\int_{-\infty}^{0} \left(Ae^{\kappa x}\right)^2 dx + \int_{0}^{\infty} \left(Ae^{-\kappa x}\right)^2 dx = 1 \Rightarrow \left[\frac{A^2}{2\kappa}e^{2\kappa x}\right]_{-\infty}^{0} + \left[\frac{A^2}{2\kappa}e^{-2\kappa x}\right]_{0}^{\infty} = 1$$

$$\Rightarrow \frac{A^2}{2\kappa} + \frac{A^2}{2\kappa} = 1$$

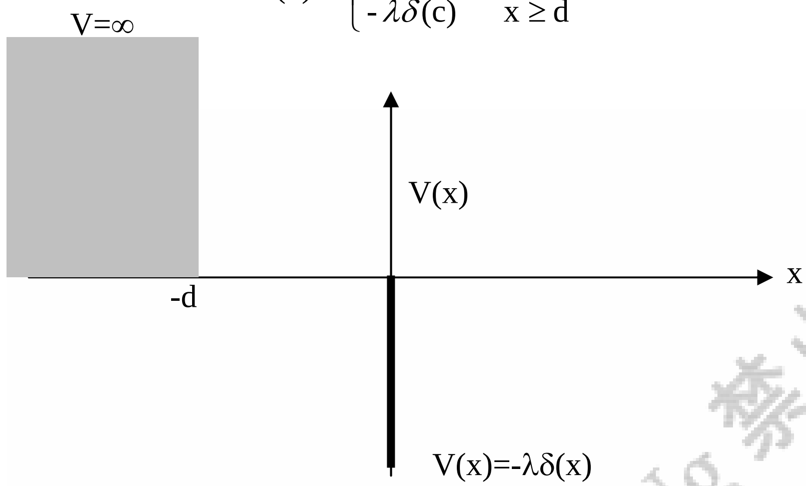
$$\Rightarrow A = \sqrt{\kappa} = \frac{\sqrt{m\lambda}}{\hbar}$$

There is always one bound state for single delta function potential.

(c) (20 points)

A wall is now placed at x=-d so that the potential is given by

$$V(x) = \begin{cases} \infty & x < -d \\ -\lambda\delta(c) & x \geq d \end{cases}$$



Solve for the eigenfunction(s) and the corresponding energy eigenvalue(s) for the bound state(s). (To save time, you do not need to normalize the state function). Can you reduce your result to that of part (b) by letting d →∞ ?

Since $E < 0$, let $\kappa = \sqrt{\dfrac{2m\,|E|}{\hbar^2}}$

The wave function for $x < 0$ is $\psi_< = Ae^{\kappa x} + Be^{-\kappa x}$ (since now x will not equal to $-\infty$)

The wave function for $x > 0$ is $\psi_> = Ce^{-\kappa x}$

$\psi(-d) = 0 \Rightarrow Ae^{-\kappa d} + Be^{\kappa d} = 0 \Rightarrow A = -Be^{2\kappa d}$ --- (1)

$\psi(-\varepsilon) = \psi(\varepsilon) \Rightarrow A + B = C \Rightarrow C = (1 - e^{2\kappa d})B$ --- (2)

Therefore the wavefunction is $\psi(x) = \begin{cases} 0 & x \leq -d \\ B(-e^{2\kappa d}e^{\kappa x} + e^{-\kappa x}) & -d < x \leq 0 \\ (1 - e^{2\kappa d})Be^{-\kappa x} & d \geq 0 \end{cases}$

B is to be determined by normalization condition :

$\int \psi^*\psi \, dx = 1$

Schroedinger equation :

$$\frac{\hbar^2}{2m}\left[\psi'(\varepsilon) - \psi'(-\varepsilon)\right] + \lambda\psi(0) = 0 \Rightarrow \frac{\hbar^2}{2m}\left[-B\kappa(1 - e^{2\kappa d}) - B(-\kappa e^{2\kappa d} - \kappa)\right] + \lambda(1 - e^{2\kappa d})B = 0$$

$$\Rightarrow \frac{\hbar^2}{2m}\left[2\kappa e^{2\kappa d}\right] + \lambda(1 - e^{2\kappa d}) = 0$$

$$\Rightarrow \kappa = \frac{m}{\hbar^2}\lambda(e^{2\kappa d} - 1)e^{-2\kappa d}$$

$$\Rightarrow \kappa = \frac{m\lambda}{\hbar^2}(1 - e^{-2\kappa d})$$

As $d \to \infty$, $e^{-2\kappa d} \to 0$ and $\kappa \to \frac{m\lambda}{\hbar^2}$, which is the same result as part (b).